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Intermediate Asymptotics and Renormalization Group Theory[1]

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The principles of the renormalization group (RG) are presented pedagogically from the point of view of intermediate asymptotics (IA), which is familiar to hydrodynamicists and applied mathematicians. To demonstrate the equivalence of RG and IA approaches, a typical statistical mechanical problem, conventionally studied by the renormalized perturbation approach, is reconsidered from the IA point of view, and renormalized perturbation theory is applied to a partial differential equation conventionally studied by IA. This example is important because it is an explicit demonstration that the RG can be applied to partial differential equations without adding a noise source. We suggest that the ideas explained in this article may be applicable to the Navier–Stokes equation.

KEY WORDS: Renormalization group; intermediate asymptotics; nonlinear parabolic equations.

1. INTRODUCTION

Attempts to apply perturbative renormalization group (RG) approaches to the problem of turbulence have apparently met with quantitative success (Yakhot and Orszag, 1986; Forster *et al.*, 1977; De Dominicis and Martin, 1979). For example, Yakhot and Orszag have successfully calculated the Kolmogorov constant for the inertial range spectrum, the turbulent Prandtl number for high-Reynolds-number heat transfer, etc. It is still fair to say that the reason for this success has not been well understood, especially since the unperturbed state for the adopted perturbation scheme and

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the state in which we are interested have a qualitatively different character. The successful applications of the renormalization group in other areas of physics have always started from states qualitatively similar to the actual physical states of interest.

Our purpose in writing this article is threefold: Firstly, we wish to explain the general approach of renormalization group methods (Wilson, 1971; Brezin *et al.*, 1976) to extract macroscopic phenomenology from microscopic models. Secondly, we point out the equivalence of RG theory and the theory of intermediate asymptotics (IA) (Barenblatt, 1979) familiar to hydrodynamicists and applied mathematicians; in so doing we present a new point of view of RG theory which we hope will be useful in the application of RG to nonlinear problems such as turbulence. We will find, *inter alia*, that the so-called anomalous dimensions in RG theory are the nontrivial exponents appearing in IA. To make our point of view as perspicacious as possible we have chosen to discuss two problems: a statistical mechanical problem, conventionally treated by RG, is solved from the IA point of view, and a problem in IA, discussed by Barenblatt (1979), is solved with the aid of RG. Thus, our work extends the range of problems that may be solved using RG. Thirdly, in the only existing RG approach to turbulence (Yakhot and Orszag, 1986; Forster *et al.*, 1977; De Dominicis and Martin, 1979), the starting point is to supplement the Navier–Stokes equation with a judiciously chosen noise source. We show, by explicit example, that even perturbative RG methods can be directly applied to partial differential equations without requiring the additional stochastic element.

This is partly a pedagogical article, but the relation between RG and IA is pointed out for the first time in the work reported here. The explicit calculations that we report are presented in more detail elsewhere (Goldenfeld *et al.*, 1989).

In Section 2, we summarize the theory of intermediate asymptotics. In Section 3, the fundamental idea of renormalization group theory as a method to extract macroscopic phenomenology from microscopic models is explained. In Section 4, a single-chain polymer, which is conventionally successfully studied by a renormalized perturbation theory, is studied from the IA point of view. In Section 5, a typical IA problem, Barenblatt's problem, is solved by a renormalized perturbation method. Section 6 is a discussion and summary.

2. A SUMMARY OF INTERMEDIATE ASYMPTOTICS

Buckingham's Π-theorem (Buckingham, 1914) states that any consistent physical relation can be written in terms of dimensionless

quantities. Consider a situation in which a dimensionless quantity Π is given as a function of a finite set of dimensionless variables (parameters) $\{\Pi_0, \Pi_1, ..., \Pi_n\}$:

$$\Pi = f(\Pi_0, \Pi_1, ..., \Pi_n) \tag{2.1}$$

The dimensionless variables Π_i are functions of dimensional quantities which are the physical variables of the problem. Suppose we are interested in the intermediate asymptotic limit of $\Pi_0 \to 0$, which is assumed to be realized by changing an appropriate physical variable in the problem. A system is said to be in an *intermediate asymptotic* state when its behavior is independent of the details of the initial and/or boundary conditions, yet has not attained its final state. In the language of statistical physics, such a regime is known as a *scaling regime*.

Barenblatt (1979) classifies the situation into three categories:

(i) The theory is well defined in this limit. That is, f is nonsingular at $\Pi_0 = 0$, and $f(0, \Pi_1, ..., \Pi_n)$ is well defined. This case is referred as *self-similarity of the first kind.*

(ii) $f(0, \Pi_1, ..., \Pi_n)$ is not well defined, but there exists a set of real numbers $\{\alpha, \alpha_1, \alpha_2, ..., \alpha_n\}$ such that the following limit is well defined:

$$\lim_{\Pi_0 \to 0} \Pi/\Pi_0^{\alpha} = \lim_{\Pi_0 \to 0} f(\Pi_0, \Pi_1/\Pi_0^{\alpha_1}, ..., \Pi_n/\Pi_0^{\alpha_n})/\Pi_0^{\alpha} \tag{2.2}$$

If f obeys a partial differential equation (PDE), then the parameters $\alpha, \alpha_1, ..., \alpha_n$ can in principle be determined from the PDE. In this formula, the exponents $\alpha, \alpha_1, \alpha_2, ..., \alpha_n$ cannot be determined by dimensional analysis. We shall see that they are nothing other than the so-called anomalous dimensions in RG theory. Barenblatt refers to this case as *self-similarity of the second kind.*

(iii) None of the above. In this case, perhaps more complicated combinations of dimensionless parameters than in (ii) are required for the limit $\Pi_0 \to 0$ to be well defined. This is a very interesting possibility, but has never been studied as far as we are aware. If $n \to \infty$, then so-called multifractal behavior falls under this category (Halsey *et al.*, 1986; Jensen *et al.*, 1987; for a more general viewpoint, see Oono, 1990).

3. PHENOMENOLOGY AS A CONSEQUENCE OF RENORMALIZABILITY

Our goal is to determine the large-scale or long-time behavior of a given system. Experience has shown that in this limit, the full complexities

of the microscopic physics can often be subsumed into a small number of phenomenological parameters. When this is indeed the case, then it is meaningful to consider a macroscopic phenomenological description. To illustrate what we mean by a macroscopic phenomenological description, consider the motion of a three-dimensional macroscopic fluid. We know, at least for small Reynolds number [empirically and mathematically (Ladyzhenskaya, 1969; Temam, 1977; Constantin and Foias, 1988)], that the macroscopic description is given by the Navier–Stokes equation. The most important fact about the Navier Stokes equation is that it describes any Newtonian fluid, and specific properties of the fluid may be expressed by only a few materials constants such as viscosity, density, etc. Thus the macroscopic phenomenological description of Newtonian fluids consists of two parts: the universal structure, i.e., the structure of the equation itself, and phenomenological parameters sensitive to the specific microscopic physics of the system. Any good phenomenological description of a system always has this structure: a universal part and a few detail-sensitive parameters (called materials constants or phenomenological parameters). In this sense, it is possible that there is no good macroscopic phenomenology of two-dimensional Newtonian fluids, even though the two-dimensional Navier–Stokes equation is well defined (Ladyzhenskaya, 1969; Temam, 1977; Constantin and Foias, 1988), because the presence of long-time tail (see, e.g., Pomeau and Resibois, 1975) renders the existence of phenomenological transport coefficients questionable.

Important examples of macroscopic phenomenology can be found in the theory of critical phenomena (see, e.g., Stanley, 1971) and polymer systems (de Gennes, 1980). The correlation length ξ of a system near a critical point is related to the temperature T as $\xi \simeq \xi_0 |(T - T_c)/T_c|^{-\nu}$, where T_c is the critical temperature, ξ_0 is a microscopic length, and the exponent ν seems only to depend on generic properties of systems, such as the dimension, or the symmetry group. Thus, in this case, T_c and the proportionality constant ξ_0 are the phenomenological parameters, and the algebraic functional form with exponent ν is the universal structure. The mean square end-to-end distance $\langle R^2 \rangle$ (average square radius) of a polymer chain with degree of polymerization N is given by $\langle R^2 \rangle \simeq \xi_0^2 N^{2\nu}$, and again the exponent ν is independent of the chemical details of the polymer and its solvent, so long as N is sufficiently large. The universal laws mentioned above are reminiscent of the universal power law of the energy spectrum E_k of fully developed turbulence: $E_k \sim k^{-\nu}$ with $\nu \simeq 5/3$ (see, e.g., Rose and Sulem, 1978). The exponent ν is presumably independent of the details of the stirring. Notice, however, that there is a fundamental difference between the statistical mechanical examples above and the problem of turbulence. In the former, specific system details, in

which we are not interested, are in the microscopic scale, but in the latter, these irrelevant details are at the largest scale of the system.

In order to extract the universal features of a system, we exploit the tautological fact that the universal structure does not depend on microscopic details in the limit of large-scale observation, i.e., the limit of (macrolength scale)/(microlength scale) $\to \infty$. Thus if we consider a set of transformations that alters only the microscopic parameters of a model (e.g., through redefining the lattice spacing of the underlying lattice or the size of the molecular unit of a polymer chain), the macroscopic universal features should remain unchanged. Therefore, if we can absorb the changes caused by modification of microscopic parameters into a few phenomenological parameters, we can obtain universal relations between phenomenological parameters. If this is possible by introducing a finite number of phenomenological parameters, we say that the model (or the system) is *renormalizable*. This is the standard method of formulating the problem of extracting macroscopic phenomenology with RG. RG seeks the microscopic detail sensitive parts in the theory and tries to absorb them into macroscopic phenomenological parameters.

Let us briefly discuss the relationship between models of physical phenomena and reality. Suppose that the macroscopic phenomenology of a system can be described successfully with a renormalizable microscopic model. The phenomenological parameters must be provided from either experiment or from a description valid at a smaller length scale. Is this a fundamental limitation of the renormalizable theory? If one is a reductionist, the answer is probably yes. However, another point of view is that microscopic models are not more fundamental than macroscopic phenomenology. In fact, it is inevitable that in constructing models of physical systems, phenomena beyond some energy scale (or on length scales below a threshold) are neglected. In this sense, all present-day theoretical physics is macroscopic phenomenology.

This discussion of the relationship between microscopic and macroscopic models has an important corollary. A macroscopic description of a system may be insensitive to microscopic details. Thus there is no unique microscopic picture that is consistent with the macroscopic phenomenology. In fact, there is an equivalence class of microscopic models all of which have the same macroscopic phenomenology. One could say that this fact is epistemologically important, but this is also very useful in practice. If one is solely interested in studying universal, macroscopic properties of a system, one can choose the most convenient or simplest (in some sense) microscopic model (called a *minimal model*).

The RG is a practical implementation of these ideas. It is a set of transformations between the parameters describing a system at a small

length scale and those describing the system on a larger scale. It is not, in principle, necessary that the transformation relate different length scales. In an example we mention later, the transformation is in time rather than in space. Obtaining the transformation is often quite difficult, and a variety of techniques have been developed by statistical physicists to construct RG transformations in the case of critical phenomena and statistical field theory.

Once an RG transformation has been constructed, the fixed points are sought. The possible fixed points as the RG is iterated to longer and longer length scale correspond to the possible macroscopic phenomenologies. Specific methods are available for studying systems near fixed points, but these are not discussed here. In the parameter space corresponding to all microscopic models, there are several basins of attraction: all microscopic models initially in a given basin of attraction flow under the RG towards the same fixed point and hence the same macroscopic phenomenology. Renormalization group theory has taught us how to extract definite macroscopic conclusions from this vague description. Of course, this is not always possible, as we have suggested in the case of two-dimensional hydrodynamics. However, we clearly recognize general macroscopic features of the world in our daily lives as macroscopic creatures! Thus, we may believe that for many important aspects of the macroscopic world there must be renormalizability. We may say that renormalizability makes physics possible.

4. A STATISTICAL MECHANICAL EXAMPLE

Renormalization group theory has been unquestionably successful in various statistical mechanical problems such as second-order phase transitions (see, e.g., Wilson and Kogut, 1974), and polymer solutions (Oono, 1985). Using an example, we outline how such problems can be considered as IA problems. Then, we reinterpret this using the standard RG approach. We use a polymer chain as a typical example. This is because the theory of polymer systems is one of the clearest applications of RG; even with the brutal approximation necessary for the calculation, the predictions for universal properties are in quite good agreement with experiment. Furthermore, the authors have some experience with this system, and given the polymer-turbulence analogy proposed by Chorin (1988), this topic should not be alien to hydrodynamicists. The approach we explain in the following is not the Wilson-type RG theory adopted by Yakhot and Orszag (1986), but much closer to that used in high-energy physics, and developed by Gell-Mann and Low (1954) (see also Bogoliubov and Shirkov, 1959).

4.1. Example—A Single-Chain Polymer

Let us consider a polymer chain suspended in a solvent. We assume that the chain has the total length N_0, which is proportional to the number of monomers in the chain, and that there is a short-range repulsive self-interaction among monomers. The monomer–monomer interaction energy for a chain, divided by $k_B T$, where k_B is the Boltzmann constant and T is the absolute temperature, can be written as

$$\tfrac{1}{2} v_0 \int d^d \mathbf{r}\, \rho(\mathbf{r})^2 \tag{4.1}$$

where ρ is the monomer density, v_0 is the interaction parameter (>0), and d is the spatial dimensionality. Monomers cannot occupy the same point in space. Hence the interaction should not contain unphysical self-interactions. Thus, *along* the chain around each point on the chain, there is a small zone of size a with which the point cannot interact. The size of this zone need not correspond to the monomer size. Our microscopic model contains three parameters, N_0, v_0, and a. We want to find the relation between the root-mean-square end-to-end distance $\langle R^2 \rangle$ of the chain in terms of these parameters.

Let us cast the problem in the language of intermediate asymptotics (IA). First, we write the equation corresponding to (2.1). We may consider the chain without self-interaction as the trajectory of a random walker up to N_0 steps. This implies that the polymer size is proportional to $\sqrt{N_0}$. That is, the length *along* the chain has effectively the dimension of (length)^2. Since a is measured *along* the chain this also should have the same dimension. For later convenience we introduce a phenomenological parameter L, which may be interpreted as the *square* of a phenomenological length scale. Hence, $[L] = [\langle R^2 \rangle] = [N_0] = [a]$. Henceforth, $[\bullet]$ denotes the dimension of $\bullet$. Since energy divided with $k_B T$ is dimensionless, (4.1) implies that $[v_0][L]^{d/2}([N_0]/[L]^{d/2})^2 = 1$ or $[v_0 L^{\varepsilon/2}] = 1$, where $\varepsilon \equiv 4 - d$. Thus the dimensionless quantities we need are

$$\Pi \equiv \langle R^2 \rangle / L, \qquad \Pi_0 \equiv a/L, \qquad \Pi_1 \equiv N_0/L, \qquad \Pi_2 \equiv v_0 L^{\varepsilon/2} \tag{4.2}$$

The equation of type (2.1) reads:

$$\Pi = f(\Pi_0, \Pi_1, \Pi_2) \tag{4.3}$$

We are interested in the limiting behavior of a very long chain, so that we wish to take the limit $\Pi_0 \to 0$. Since this limit could also be realized by $a \to 0$, unphysical self-interactions cause the model to be ill defined. The theory actually contains divergences in the $\Pi_0 \to 0$ limit. Therefore, we can-

not have the situation (i) referred to in Section 2. Experimentally, we know (e.g., Daoud *et al.*, 1975) that there is a well-defined phenomenological description in terms of the effective polymer length N and the effective interaction parameter v. This means that our problem is in category (ii). Hence, there must be numbers α, α_1, and α_2 as in (2.2) such that

$$\lim_{a \to 0} \Pi/\Pi_0^{\alpha} = \lim_{a \to 0} f(\Pi_0, \Pi_1/\Pi_0^{\alpha_1}, \Pi_2/\Pi_0^{\alpha_2})/\Pi_0^{\alpha} \tag{4.4}$$

is well defined. To have the well-defined phenomenology which we know empirically in the present example, we must be able to absorb a into phenomenological parameters. This becomes possible if we interpret phenomenological parameters as follows:

$$N = N_0 \Pi_0^{\alpha_1}, \qquad v = v_0 \Pi_0^{\alpha_2} \tag{4.5}$$

The exponent α must be zero, because $\langle R^2 \rangle$ is a directly observable quantity, so that it cannot be redefined. In contrast, v_0 and N_0 are parameters that we cannot observe directly by macroscopic observation. This is why we can introduce N and v as macro-observables.

In the renormalization group approach we introduce renormalization constants Z_1 and Z_2 as

$$\langle R^2 \rangle = Lf(v_0 L^{\varepsilon/2} Z_1, (N_0/L)\, Z_2) \tag{4.6}$$

We define phenomenological (or renormalized) parameters as

$$N = N_0 Z_1, \qquad v = v_0 Z_2 \tag{4.7}$$

and the divergences due to $a \to 0$ limit are absorbed into these redefined parameters. Thus we have a perfect parallelism between RG and IA. Comparing (4.5) and (4.7), we should have

$$Z_1 \sim \left(\frac{L}{a}\right)^{\alpha_1}, \qquad Z_2 \sim \left(\frac{L}{a}\right)^{\alpha_2} \tag{4.8}$$

This is indeed the case. We compute $\langle R^2 \rangle$ perturbatively in $u_0 \equiv v_0 L^{\varepsilon/2}$ (or equivalently in $u \equiv v L^{\varepsilon/2}$), absorbing the divergences in the $a \to 0$ limit order by order into Z_1 and Z_2. Thus we can compute the α's perturbatively. We will demonstrate this method later with Barenblatt's example. In the present case (Ohta *et al.*, 1982) $\alpha_1 = -u/\pi^2$ and $\alpha_2 = u/4\pi^2$ to the lowest nontrivial order.

4.2. Renormalization Group Equation

An important consequence of the assertion that there is a macroscopic phenomenological description, in the sense discussed in Section 3, is the renormalization group equation. The key point is that changing L should not change macroscopic observables. For example, $\langle R^2 \rangle$ should be a definite number independent of L, if a microscopic model is fixed, because L is introduced independent of the microscopic model. Hence, we have an obvious identity:

$$L \left. \frac{\partial \langle R^2 \rangle}{\partial L} \right|_{\text{microparameters}} = 0 \tag{4.9}$$

Let us introduce a function F:

$$\langle R^2 \rangle / L = F(u, X) \tag{4.10}$$

where u is already introduced, and $X = N/L$. Using the chain rule, we can compute this identity more explicitly as

$$F + \beta(u) \frac{\partial F}{\partial u} + [\gamma(u) - 1] X \frac{\partial F}{\partial X} = 0 \tag{4.11}$$

where

$$\beta(u) \equiv \frac{\partial u_0}{\partial u}, \qquad \gamma(u) \equiv \frac{\partial \ln Z_2}{\partial L} \tag{4.12}$$

(4.11) is called the *renormalization group equation* (for $\langle R^2 \rangle$). The derivatives are computed with fixed microscopic parameters. To fully utilize this equation we need β to order u^2: it reads (Ohta *et al.*, 1982; Oono, 1985)

$$\beta(u) = u(u^* - u)/\pi^2 \tag{4.13}$$

with $u^* = \pi^2 \varepsilon / 2$ to the nontrivial lowest order. γ is obtained from (4.8) as $\gamma(u) = u/(2\pi)^2$ to this order. Solving (4.12), it is easy to demonstrate that for large N (or X) we may set $u = u^*$. Therefore, for simplicity, here we set $\beta = 0$ and $\gamma = \gamma^* \equiv \varepsilon/8$ in (4.12). We have only to solve the following ordinary differential equation:

$$F + (\gamma^* - 1) X \frac{dF}{dX} = 0 \tag{4.14}$$

Thus, we get

$$\langle R^2 \rangle = AL\left(\frac{N}{L}\right)^{1/(1-\nu^*)} \tag{4.15}$$

where A is a nonuniversal constant. If we know that the problem is in category (ii), then we may demand that the renormalization group equation like (4.9) holds asymptotically. With the aid of the RG equation, we can compute nontrivial exponents perturbatively. The nature of the resultant expansion formulas is generally unclear, but they are believed to be Borel summable asymptotic expansions.

5. A PDE EXAMPLE—BARENBLATT'S PROBLEM

Barenblatt (1979) discusses the following nonlinear parabolic equation:

$$\frac{\partial u}{\partial t} = \begin{cases} \frac{1}{2}\frac{\partial^2 u}{\partial x^2} & \text{if} \quad \partial_t u > 0 \\ \frac{1}{2}(1+\varepsilon)\frac{\partial^2 u}{\partial x^2} & \text{if} \quad \partial_t u < 0 \end{cases} \tag{5.1}$$

where $x \in (-\infty, +\infty)$, $t \in (0, +\infty)$, and the initial condition is given by

$$u(x, 0) = g(x) \equiv \frac{Q_0}{(2\pi\delta)^{1/2}} e^{-x^2/2\delta} \tag{5.2}$$

where δ is a positive number.

5.1. Intermediate Asymptotics Approach

We want to study the solution in the small δ limit, or in the $t \to \infty$ limit. The problem contains variables x, t, and parameters ε, Q_0, and δ. The last parameter corresponds to the cutoff parameter in the preceding section. Analogous to L in the preceding section, it is convenient to introduce a phenomenological time scale T, which defines the macroscopic unit of time. Thus, we have the following dimensionless quantities:

$$\Pi = u\sqrt{T}/Q_0, \qquad \Pi_0 \equiv \delta/T, \qquad \Pi_1 \equiv x^2/T, \qquad \Pi_2 \equiv t/T, \qquad \Pi_3 = \varepsilon \tag{5.3}$$

Barenblatt demonstrates that this is actually a category (ii) problem in the $\delta \to 0$ limit. In this case, x and t are directly observable, so α_1 and α_2

should be zero. If we interpret (5.1) as a macroscopic equation for long time behavior, then ε should also be directly observable, so that α_3 must vanish as well. The only quantity we cannot know from macroscopic observation is Q_0, so that the equation corresponding to (2.2) should read

$$\lim_{\delta \to 0} \Pi/\Pi_0^{\alpha} = \lim_{\delta \to 0} f(\Pi_0, \Pi_1, \Pi_2, \Pi_3)/\Pi_0^{\alpha} \tag{5.4}$$

This implies that we may asymptotically assume the following functional form for the solution:

$$u(x, t) = t^{-(1/2+\alpha)} f(x^2/t) \tag{5.5}$$

Barenblatt introduces this into the original equation (5.1) and obtains a nonlinear eigenvalue problem for α, which cannot be solved analytically.

5.2. Renormalized Perturbation Approach

Now we present our solution to this problem from the statistical physics point of view. First, we construct a formal solution to (5.1) as follows:

$$u(x, t) = \int dy\, G(x-y, t)\, g(y) + \tfrac{1}{2}\varepsilon \int_0^t ds \int dy\, G(x-y, t-s)\, \theta[-\partial_s u(y, s)]\, \partial_y^2 u(y, s) \tag{5.6}$$

where θ is the Heaviside step function and G is the following Green's function:

$$G(x, t) = \frac{1}{(2\pi t)^{1/2}} e^{-x^2/2t} \tag{5.7}$$

Next we solve this equation perturbatively in powers of ε:

$$u = u_0 + \varepsilon u_1 + \cdots \tag{5.8}$$

The zeroth order is given by

$$u_0 = \frac{Q_0}{[2\pi(t+\delta)]^{1/2}} \exp\left(-\frac{x^2}{2(t+\delta)}\right) \tag{5.9}$$

In the computation of the first order we use the zeroth-order solution in the step function. We find that

$$u_1 = \frac{Q_0}{4\pi} \int_{\delta}^{t+\delta} ds \int_{-\sqrt{s}}^{+\sqrt{s}} dy\, \frac{1}{s}\left(\frac{y^2}{s} - 1\right) \times \frac{1}{[s(t-s-\delta)]^{1/2}} e^{-y^2/2s} e^{-(x-y)^2/2(t-s-\delta)} \tag{5.10}$$

We are interested in the $\delta \to 0$ limit. Replacing $t+\delta$ with t in (5.10), we get a formula correct to order δ^0:

$$u_1 = \frac{Q_0}{4\pi}\int_\delta^t ds \int_{-\sqrt{s}}^{+\sqrt{s}} dy \frac{1}{s}\left(\frac{y^2}{s}-1\right) \times \frac{1}{[s(t-s)]^{1/2}} e^{-y^2/2s} e^{-(x-y)^2/2(t-s)} + O(\delta) \tag{5.11}$$

We can isolate the singularity in the $\delta \to 0$ limit explicitly:

$$u_1 = \frac{Q_0}{4\pi}\frac{1}{\sqrt{t}} e^{-x^2/2t}\int_\delta^t ds \frac{1}{s}\int_{-1}^{+1} dw\, e^{-w^2/2}(w^2-1) + \frac{Q_0}{4\pi}\int_0^t ds \int_{-1}^{+1} dw \frac{1}{s}\left[\frac{1}{(t-s)^{1/2}} e^{-(x-\sqrt{s}w)^2/2(t-s)} - \frac{1}{\sqrt{t}} e^{-x^2/2t}\right] \times e^{-w^2/2}(w^2-1) + O(\delta) \tag{5.12}$$

The first term in (5.12) is singular in the small δ limit:

$$[u_1]_{\text{singular}} = -\frac{1}{(2\pi e)^{1/2}} \ln(t/\delta) \times u_0 \tag{5.13}$$

Thus, we have obtained

$$u(x,t) = \frac{Q_0}{(2\pi t)^{1/2}} e^{-x^2/2t}\left(1 - \varepsilon \frac{1}{(2\pi e)^{1/2}} \ln\frac{t}{\delta} + \cdots\right) + (\text{nonsingular terms}) \tag{5.14}$$

This is the so-called "bare" perturbation result. Now, we must renormalize it; that is, we must try to absorb this singularity into a phenomenological parameter.

According to our discussion at the beginning of this section, Q_0 is the "microscopic parameter" we cannot know directly from a large-scale observation. Hence, we may introduce a phenomenological counterpart Q, and the accompanying renormalization constant Z as

$$Q = ZQ_0 \tag{5.15}$$

The renormalization constant Z is dimensionless, so that it can depend only on dimensionless constants $\Pi_0 \equiv \delta/T$ and $\Pi_3 \equiv \varepsilon$. We assume that Z can be expanded in powers of ε as

$$Z = 1 + \varepsilon A(T/\delta) + \cdots \tag{5.16}$$

and try to absorb the singularity in (5.14) order by order. This is exactly the method by which (4.8) was obtained. Splitting the singular term in (5.14) using

$$\ln \frac{t}{\delta} = \ln \frac{T}{\delta} + \ln \frac{t}{T} \tag{5.17}$$

we can rewrite (5.14) as

$$u(x, t) = \frac{Q_0}{(2\pi t)^{1/2}} e^{-x^2/2t} \left(1 - \varepsilon \frac{1}{(2\pi e)^{1/2}} \ln \frac{T}{\delta} + \cdots \right) + (\text{nonsingular terms}) \tag{5.18}$$

Putting (5.16) into (5.18), we get

$$u = \frac{Q}{[2\pi(t)]^{1/2}} e^{-x^2/2t} \left\{ 1 - \varepsilon \left[\frac{1}{(2\pi e)^{1/2}} \ln \frac{T}{\delta} + A \right] + \cdots \right\} + (\text{nonsingular terms}) \tag{5.19}$$

Therefore the following choice allows us to absorb the singularity into Q:

$$A = -\frac{1}{(2\pi e)^{1/2}} \ln \frac{T}{\delta} \tag{5.20}$$

Proceeding naively, we expect that

$$Z = \left(\frac{T}{\delta} \right)^{\varepsilon/(2\pi e)^{1/2} + \cdots} = \Pi_0^{-\alpha} \tag{5.21}$$

where we have used the definition of α in (5.4). Hence, we have obtained $\alpha = \varepsilon/(2\pi e)^{1/2} + \cdots = 0.24197 \cdots \varepsilon$. This indeed agrees with the expansion (Liu, 1989; Goldenfeld *et al.*, 1989) obtained from the analytical result due to Barenblatt (1979). The final result in the $\delta \to 0$ limit reads

$$u(x, t) = \frac{Q}{(2\pi t)^{1/2}} e^{-x^2/2t} + \varepsilon \times (\text{nonsingular terms}) \tag{5.22}$$

This is called the renormalized perturbation series for u.

5.3. Renormalization Group Equation

In the above computation of α, we have assumed the power law form for Z. This is justified by the renormalization group equation akin to (4.9). Let us write

$$u = \frac{Q_0}{\sqrt{T}} Z f \left(\frac{x^2}{t}, \tau, \varepsilon \right) \tag{5.23}$$

where $\tau \equiv t/T$. The RG equation reads

$$T \left. \frac{\partial u}{\partial T} \right|_{x,t,\varepsilon,\delta \to 0} = 0 \tag{5.24}$$

Using the chain rule, we can rewrite this as

$$-\frac{1}{2} f - \alpha f - \tau \frac{\partial f}{\partial \tau} = 0 \tag{5.25}$$

where α is defined as

$$\alpha \equiv -\frac{\partial \ln Z}{\partial \ln T} \tag{5.26}$$

Solving (5.25), we get the following general solution:

$$f = \left(\frac{T}{t}\right)^{\alpha + 1/2} F\left(\frac{x^2}{t}, \varepsilon\right) \tag{5.27}$$

where F is a function yet to be determined. Hence, we have

$$u = \frac{T^{\alpha} Q}{t^{1/2+\alpha}} F\left(\frac{x^2}{t}, \varepsilon\right) \tag{5.28}$$

The unknown function F can also be determined perturbatively through comparison of (5.28) and the renormalized perturbation result (5.22). We will not discuss this straightforward but often tedious process in this article. An important point is that we can often improve perturbational results considerably with the aid of the RG equation. One of the most successful examples can be seen in the statistical physics of polymer solutions (Oono, 1985).

6. DISCUSSION

In this article we have explained the principles of renormalization group (RG) theory as a general framework for extracting phenomenological relations among macroscopic observables from microscopic models that may not be precisely definable. Thus, RG approaches should not be restricted to statistical mechanical and field theoretical problems only. This point of view has enabled us to discover the equivalence of the theory of intermediate asymptotics (IA), which is traditionally applied to partial differential equations (PDE), and RG theory. We hope that this relation will make

where $\tau \equiv t/T$. The RG equation reads

$$T \frac{\partial u}{\partial T}\bigg|_{x,t,\varepsilon,\delta \to 0} = 0 \tag{5.24}$$

Using the chain rule, we can rewrite this as

$$-\frac{1}{2} f - \alpha f - \tau \frac{\partial f}{\partial \tau} = 0 \tag{5.25}$$

where α is defined as

$$\alpha \equiv -\frac{\partial \ln Z}{\partial \ln T} \tag{5.26}$$

Solving (5.25), we get the following general solution:

$$f = \left(\frac{T}{t}\right)^{\alpha + 1/2} F\left(\frac{x^2}{t}, \varepsilon\right) \tag{5.27}$$

where F is a function yet to be determined. Hence, we have

$$u = \frac{T^{\alpha} Q}{t^{1/2 + \alpha}} F\left(\frac{x^2}{t}, \varepsilon\right) \tag{5.28}$$

The unknown function F can also be determined perturbatively through comparison of (5.28) and the renormalized perturbation result (5.22). We will not discuss this straightforward but often tedious process in this article. An important point is that we can often improve perturbational results considerably with the aid of the RG equation. One of the most successful examples can be seen in the statistical physics of polymer solutions (Oono, 1985).

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In this article we have explained the principles of renormalization group (RG) theory as a general framework for extracting phenomenological relations among macroscopic observables from microscopic models that may not be precisely definable. Thus, RG approaches should not be restricted to statistical mechanical and field theoretical problems only. This point of view has enabled us to discover the equivalence of the theory of intermediate asymptotics (IA), which is traditionally applied to partial differential equations (PDE), and RG theory. We hope that this relation will make

possible not only a deeper understanding of RG, but also wider applications of RG methods to applied mathematics and hydrodynamic problems. We have demonstrated that a conventional perturbative scheme to implement RG can indeed solve a typical PDE problem studied by IA.

We would like to emphasize that RG is not tied to statistical systems (stochastic systems). Conventionally, RG is used to study the effect of microscopic noise, coupled to the nonlinearity of the system, on macroscopic observables. However, as has been explicitly demonstrated with the aid of a PDE example, the existence of a stochastic or statistical element is not required. Generally speaking, RG is a method to study the asymptotic behavior of a system, e.g., in the macroscopic limit, where the scale of observation is much larger than the scale of microscopic description. We have explained at length what we mean by macroscopic description of a system. It should be stressed that phenomenology is not a second-rate incomplete description of a given system, but the most complete description modulo microscopic unknown factors over which we can never have ultimate detailed knowledge.

We should also emphasize that RG methods are not restricted to perturbative approaches. This point should be clear from the existence of the so-called real-space RG methods (see, e.g., Burkhardt and van Leeuwen, 1982). This line of research should be seriously pursued to understand turbulence, but it is generally very difficult to get reliable results without extensive computational efforts.

Perturbative implimentations of RG approaches are used almost exclusively in analytical work to date. As we have already mentioned in the Introduction, in all the successful examples of perturbative RG calculations, the unperturbed system and the exact system are, in a certain sense, not as different as we might otherwise expect. For example, the shapes of a polymer chain in three-dimensional space with and without self-repulsions are, after an appropriate scaling of the sizes, not drastically different (see, e.g., Fig. 8 in Oono, 1985). The nonlinear effect, which is treated perturbatively, on the transport properties near the critical point is, after all, not extremely drastic. Orszag (1989) suggests that this is indeed the case even for turbulence.

However, we feel that there is a fundamental difference between the conventionally successful examples of perturbative RG approaches and turbulence. Let us take the simplest nontrivial example, called model A (Hohenberg and Halperin, 1977):

$$\frac{\partial \psi}{\partial t} = -(\tau\psi + g\psi^3 - D\,\Delta\psi) + \eta \tag{6.1}$$

where τ, g, and D are positive constants, Δ is the Laplacian, and η is an

appropriate spatially uncorrelated white noise. The perturbation parameter is g. This equation describes equilibrium dynamics near a critical point. The system always has only one characteristic macroscopic length ξ, the correlation length. We are interested in the large correlation length limit. We shrink the spatial scale to facilitate the study of this limit. Although this scaling makes D very small, other parameters also become very small near the critical point. Thus the solution cannot develop discontinuities or singularities as $\xi \to \infty$, because there is only a single important length scale.

Let us next consider an example which defies conventional approaches:

$$\frac{\partial \psi}{\partial t} = \Delta(-\tau\psi + g\psi^3 - D\,\Delta\psi) + \eta \tag{6.2}$$

This is called the Cahn–Hilliard–Cook equation (see, e.g., Furukawa, 1985) describing the phase separation process of binary alloys (spinodal decomposition). The crucial point is the existence of an extra Laplacian and the minus sign in front of τ. In this case, the solution has two length scales, one independent of time t, and the other scaling as $t^{1/3}$. The physical meaning of the former length scale is the thickness of the interface between separating bulk phases, and that of the time-dependent one is the size of the growing bulk phases. From the macroscopic point of view, we are interested in the limit $t \to \infty$, so that the former length scale eventually becomes infinitesimally small after rescaling the spatial variable. Therefore, the solution from the macroscopic point of view is characterized by the existence of discontinuities or shock structures. Note that the rescaling of the length scale is tantamount to reducing D with other parameters kept constant. Thus we are interested in the case with infinitesimally small D. Another such example is the Kolmogorov–Petrovsky–Piscounoff (KPP) or Fisher equation (Kolmogorov *et al.*, 1937; Fisher, 1936):

$$\frac{\partial \psi}{\partial t} = D\,\Delta\psi + \psi(1-\psi) \tag{6.3}$$

which can exhibit a propagating shock front. Again, in this case shrinking the spatial length scale is equivalent to reducing D. Notice that in these two cases, solutions even lose their differentiability in the macroscopic limit. Let us call the problems in this category shock-type.

Now, let us consider the Navier–Stokes equation. In which category is it, the shock-type or the smooth-type as model A? Obviously, the parameter corresponding to D in the above examples is the reciprocal of the Reynolds number, Re. In the $\mathrm{Re} \to \infty$ limit, the Navier–Stokes equation reduces to

the Euler equation. Hence, it is obviously in the shock-type category. Therefore, in order to understand the RG approach to the Navier–Stokes equation, we believe it is crucial to find a systematic perturbative scheme for the KPP equation, because it is the simplest shock-type equation.

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