Quasiparticle localization transition in dirty superconductors

Hyejung Kim

May 6, 2002

Abstract

This paper describes the quasiparticle localization transition within a superconducting state by field-theoretical and numerical approaches. Starting with the Bogoliubov deGennes Hamiltonian and setting models for numeric calculation, some critical properties including the localization length exponent are obtained.

1 Introduction

Anderson localization in normal disordered electronic system has been intriguing problems and recently, it has been asked whether an analogous transition can occur in the behavior of low-energy quasiparicle excitations about superconducting groud. Anderson localization will be briefly outlined next and the models used to describe quasiparticle excitations in the absence of interactions will be reviewed. And some details of the two phases, the "thermal metal"and the "thermal insulator" will be expanded. Hence using both field-theoretic and numerical methods the localization length exponent ν associated with the thermal metal-thermal insulator transition will be calculated as well as other critical exponents. One of the interesting point is that here numerical work has been significantly reduced by suitable setting transfer matrices even though it shows some gap with the field-theoretical results because of system sizes, etc. These results will be compared with experimental ones and other possibilities will be discussed.

2 Anderson Localization

In 1958, Anderson pointed out that the electric wave function in a random potential may be profoundly altered if the randomness is sufficiently strong.

According to traditional view, the Bloch waves lose phase coherence due to scattering and the wave function remains extended throughout the sample. But if the disorder is very strong, the wave function may become localized, in that the envelope of the wave function decays exponentially from some point in space, i.e.,

$$|\psi(\vec{r})| \sim \exp(|\vec{r} - \vec{r}_0|/\xi). \tag{1}$$

and ξ is the localization length. The existence of the localized state can be understood in the limit of very strong disorder. The admixture between different orbitals will not produce an extended state composed of linear combinations of infinitely many localized orbitals because the wave functions overlap significantly and admixture is small. On the other hand, states that are nearly degenerate are in general very far apart in space, so that the overlap is exponentially small. Thus, in the strongly disordered limit, the wave function will be exponentially localized.

3 Models And Symmetries

3.1 The superconducting Hamiltonian and field-theoretic formulation

The Bogoliubov deGennes Hamiltonian for quasiparticle exitations has the form below in the lattice version.

$$\mathcal{H}_{0L} = \sum_{i,j}[t_{ij} \sum_{a} c^{\dagger}_{j\alpha} c_{j\alpha} + (\Delta_{ij} c^{\dagger}_{j\uparrow} c^{\dagger}_{j\downarrow} + H.c.)] \tag{2}$$

$c^{\dagger}$ and c are electron creation and annihilation operators respectively. Defining a new set of fermionic d operators

$$d_{i\uparrow} = c_{i\uparrow}, \qquad d_{i\downarrow} = c^{\dagger}_{i\downarrow} \tag{3}$$

and from Hermiticity, the Hamiltonian of Eq. (2) can takes the form

$$\mathcal{H}_L = \sum_{i,j} d^{\dagger}_j \begin{pmatrix} t_{ij} & \Delta_{ij} \\ \Delta^*_{ij} & -t^*_{ij} \end{pmatrix} d_j \equiv \sum_{ij} d^{\dagger}_i H_{ij} d_j. \tag{4}$$

The Hamiltonian $\mathcal{H}_L$ may be diagonalized by solving the eigenvalue equation

$$\sum_i = H_{ij} \begin{pmatrix} u(i) \\ v(j) \end{pmatrix} = E \begin{pmatrix} u(i) \\ v(j) \end{pmatrix} \tag{5}$$

where u and v describe the wave-function amplitudes at each site. In gapless superconductors, states about the Fermi energy are very important to transport properties, which depend crucially on the nature of the eigenstates of Eq.(5). at $E = 0$. In particular, depending on whether these eigenstates are extended of localized, one can conceive of two very different superconducting phase-the "thermal metal" that is capable of transporting energy, and the "thermal insulator" that cannot conduct energy over large length scales.

In a field-theoretic framework, an effective action has the form below and with a scaling theory we can extract the localization length exponent and the unusual singular behavior of the density of states at the Fermi energy.

$$S_{NL\sigma M} = \int d^d x [\frac{1}{2t} \mathrm{Tr}(\vec{\nabla} U \cdot \vec{\nabla} U^\dagger) - \eta \mathrm{Tr}(U + U^\dagger)]. \tag{6}$$

3.2 Hamiltonian for numerics and the transfer matrix

In obtaining the localization length exponent and the density of states, an analog of the tight-binding Anderson model shows crisp data with much less noise than other models. With reference to Eq.(2), the couplings take the form

$$t_{ij} = \begin{cases} \frac{1}{\sqrt{2}}, & ij, \mathrm{nn} \\ V_{it}, & i = j \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

$$\Delta_{ij} = \begin{cases} \frac{1}{\sqrt{2}}, & ij, \mathrm{nn} \\ V_{i\Delta}, & i = j \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

where nn denotes nearest neighbors, and V_{it} and $V_{i\Delta}$ are on-site random variables. In a three-dimensional cubic lattice with real couplings, the Hamiltonian has the specific form

$$\mathcal{H}_L = \frac{1}{\sqrt{2}} [t \otimes \sigma^z + \Delta \otimes \sigma^x]. \tag{9}$$

The density of state can be obtained from the Hamiltonian of Eq.(9) by exact diagonalization and the localization length can be extracted by means of a transfer matrix formulation that caters to Eq.(9). The dimensions of the matrices involved in numerical work has been tremendously reduced and this is one of the different

and strong feature of this method.
To formulate the transfer matrix, consider a quasi-one dimensional strip in d dimensions with cross-sectional area L^{d-1}, and in-going and out-going states at either end of this long strip as shown in Fig. 1. The transfer matrix T can be constructed by multiplying a set of transfer matrices, each appropriate for a slice of the strip shown in Fig. 1.

$$\vec{\psi}_{i+1,R} = T_i \vec{\psi}_{i,r} \tag{10}$$

$$T = \prod_{i=1}^{N} T_i \tag{11}$$

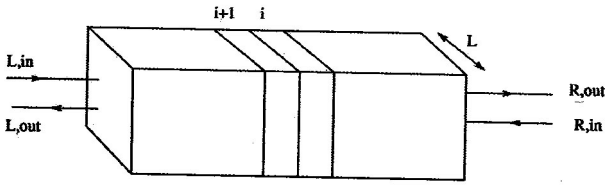

Figure 1: A quasi-one-dimensional setup for transfer matrices

In this theory the transfer matrix finally has the simple form.

$$T_i = \begin{pmatrix} B_{i,i+1}(E - A_i) & -I \\ I & 0 \end{pmatrix} \tag{12}$$

where A_i and $B_{i,i+1}$ is

$$A_i = \begin{pmatrix} t_{ii} & \Delta_{ii} \\ \Delta_{ii} & -t_{ii} \end{pmatrix} \tag{13}$$

$$B_{i,i+1} = \begin{pmatrix} t_{i,i+1} & \Delta_{i,i+1} \\ \Delta_{i,i+1} & -t_{i,i+1} \end{pmatrix}. \tag{14}$$

With this, we can extract the localization length for different values of energy E and disorder W. The energy E will be set to zero because the behavior of states at the Fermi energy will be explored.

To extract the localization length, one begins with an orthonormal basis of vectors $\hat{o}_i(0)$ in the space of the transfer matrix, representing the right-most states in Fig. 1. Assuming a set of eigenmodes $\hat{w}_i(L, W)$ that describes typical eigenmodes for a given disorder strength W, and width L, also assuming the set of basis vectors $\hat{o}_i(0)$, one obtains a set of orthogonal vectors $\vec{o}_i(1)$ with associated Lyapunov exponents

$$\gamma_i(1) = \frac{\ln |\vec{o}_i(1)|}{N}, \tag{15}$$

which give the characteristic inverse-localization lengths associated with each mode. In the quasi-one-dimensional case, all modes are exponentially decaying or growing since even slightest disorder is enough to localize states. But in the three-dimensional limit, the modes experience a transition from extended to localized behavior as a function of disorder. To determine the critical order strength W_c for this transition, a finite-size scaling analysis of the quasi-one-dimensional system can be used. The below dimensionless parameter will be used for the scaling function.

$$\Lambda(L, W) = \frac{\lambda(L, W)}{L}. \tag{16}$$

4 Critical Properties

4.1 Localization length exponent

By the finite-size scaling technique, the localization length exponent ν will be calculated here. We have

$$\frac{\lambda(L, W)}{L} = \Lambda(L, W) = h(\frac{\xi(W)}{L}). \tag{17}$$

where h is a scaling function to be determined. The localization length ξ behaves like below, close to the critical point W_c.

$$\xi \sim |W - W_c|^{-\nu}. \tag{18}$$

Λ should be well behaved and finite while the argument of x in Eq.(18) blows up at the critical point. So,

$$\lim_{x \to \infty} h(x) = \text{const.} \tag{19}$$

Then, in Eq. (17), $\ln \Lambda$ can be expressed by the function of $L^{1/\nu}$ and W_c and linearizing the function yields

$$\ln \Lambda(L, W) = \ln \Lambda_c + A(W - W_c)L^{1/\nu}. \tag{20}$$

Using an iterative procedure, data can be collapsed onto curves of $\ln \Lambda(W, L)$ vs $L^{1/\nu}(W - W_c)$ like Fig. 2. and this gives the value for the localization length exponent.

$$\nu = 1.15 \pm 0.15. \tag{21}$$

In this transfer-matrix calculations, systems have dimensions $4L^2 \times 4L^2$ with cross-sectional areas of linear dimensions $L = 4, 6, 8, 10$. A total of 2000 slices is utilized in the quasi-one-dimensional system for each value of L and W. In

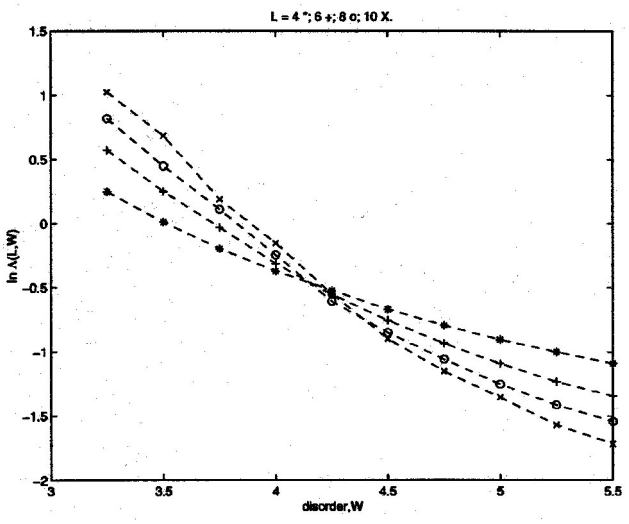

Figure 2: The scaling function $\ln \Lambda(L.W)$ as a function of disordered

comparison to the Anderson model for normal systems, this simulations require a much smaller number of transfer matrices for relatively noise-free data. This is from the relatively low critical-disorder strength.

The field theory predicts $\nu = 2/3$ versus $\nu_n = 1$ in three dimensions while numerical calculation obtains $\nu = 1.15 \pm 0.15$ versus $\nu_n = 1.54 \pm 0.08$[Ref.12] in three dimensions. This is because the system sizes and computing power were relatively low compared to the current cutting-edge procedures. A lot of work has been done to refine this.

4.2 Density of states

In superconducting systems, critical exponents can be written like below from the analogy between the field theory and the magnetic systems.

$$\rho(\Delta, E = 0) \sim |\Delta|^{\beta}, \tag{22}$$
$$\rho(\Delta = 0, E) \sim |E|^{1/\delta}, \tag{23}$$
$$\tilde{\chi} \sim |\Delta|^{-\gamma}, \tag{24}$$
$$\xi \sim |\Delta|^{-\nu}, \tag{25}$$

where ρ is the density of states, $\Delta = (W - W_c)/W_c$ and $\tilde{\chi} = d\rho/dE$. Using the free-energy density and imposing the physical constraint, the relation between critical exponents are obtained.

$$\beta = \nu(d - y), \tag{26}$$

$$\frac{1}{\delta} = \frac{d}{y} - 1, \tag{27}$$

$$\gamma = \nu(2y - d). \tag{28}$$

where y is the scaling form of E. Then, in three-dimensions approximately $y = 12/5$, $\delta = 4$, $\nu = 2/3$, $\beta = 2/5$, and $\gamma = 6/5$. This is the field-theoretical results.

To get numerical results, below scaling arguments are used.

$$\rho_L(\Delta, E = 0) = L^{-\beta/\nu}\tilde{Y}(\Delta L^{1/\nu}), \tag{29}$$

where $\tilde{Y}$ is another scaling function. Fig. 3. shows the plots ρ_L as a function of disorder for different system sizes L. To extract β, one performs a fit taking W_c, ν and β as variable parameters. By linear interpolation for a given set of system sizes and fixed arguments in $\tilde{Y}$, β can be determined. As in the localization length exponent, we can see the data collapse shown in Fig. 4, and the result is $\beta = 0.15$, $W_c = 4.67$, $\nu = 1.25$. while the field theoretic result predicts that $\beta = 0.4$. Again, as in the case of the localization length exponent, this is because of the system sizes and computing power. Still, this result shows quite clearly that the density of states at criticality exhibits a power-law suppression about the Fermi Energy, and that $\rho(E = 0, W)$ acts as an "order parameter" with a nontrivial exponent β in surprising contrast to normal systems which have $\rho(E, W_c)$ smooth about E_F, and a vanishing exponent β.

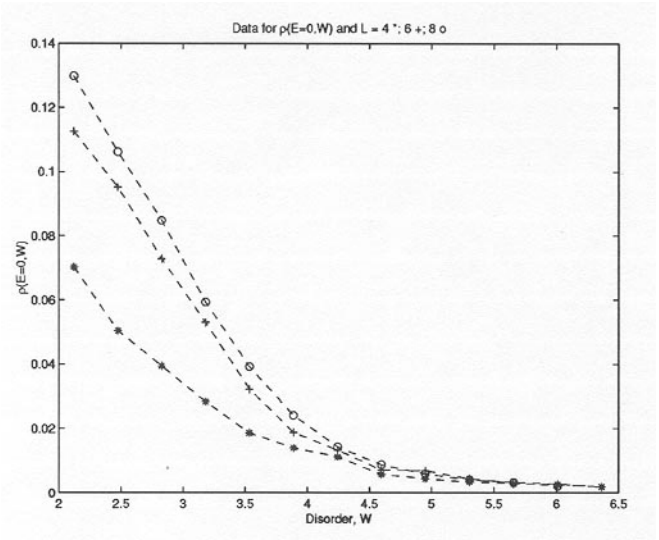

Figure 3: Density of states at $E = 0$ for various disorder strengths W

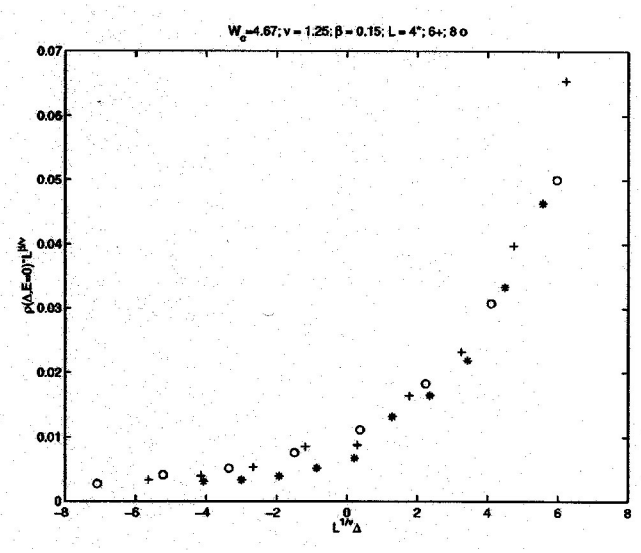

Figure 4: Data collapse of $\rho(E=0,W)$ and the critical exponent $\beta = 0.15$

5 Experiment results and other possibilities

In contrast to normal systems, the density of states of superconducting systems shows singular behavior about the Fermi energy for both phases and at the critical point. It obeys a power-law behavior of the form $\rho(E) \sim |E|^{\alpha}$, where $\alpha = \frac{1}{2}$ well within the thermal metal, $\alpha = 1$ deep in the thermal insulator, and field theory predicts $\alpha = \epsilon/4$ at the transition, with $\epsilon = 1$ for three-dimensional systems. This singular behavior ought to be reflected to in thermodynamic quantities such as specific heat and spin susceptibility, and in tunnelling experiments. The specific heat would have a form $C \sim T^{1+\alpha}$. Recent experiments of cuprate superconductors have observed a depression of the specific heat close to the Fermi energy.

C.F. Chang, J.Y.Lin and H.D. Yang have got these result. They measured low-temperature specific heat $C(T,H)$ of $La_{1.9}Sr_{0.1}Cu_{1-x}Zn_xO_4$ (x=0, 0.01, and 0.02) both in zero and applied magnetic fields. Fig. 5. shows $C(T,H=0)/T$ versus T^2 of $La_{1.9}Sr_{0.1}Cu_{1-x}Zn_xO_4$. Intriguingly, C/T of both Zn-doped samples shows a dip at low temperatures, most evidently below 2K, while this dip is absent in the x=0 sample. This dip is quite peculiar as we can see comparing to data of Copper(dash line) in Fig. 5. C/T of Copper shows a straight line in this temperature range.

They also measured the magnetic field dependence of $C(T,H)$. For $x = 0.01$ sample, with increasing H, depression of the specific heat at low temperatures becomes more significant, and the temperature T_0 below which the depression occurs, increases as shown in Fig. 6.

To compare the data with the theory, $C(T,H)$ below 2K has been analyzed based on the model $\rho(E) = \rho(0) + A\rho^{\alpha}$. This analysis leads to a non-zero $\rho(0)$ which is further depressed by H. Meanwhile, it is found that $\alpha = 1/2$ gives a

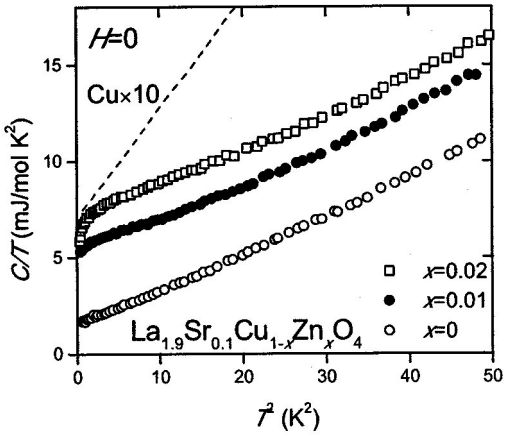

Figure 5: C/T vs. T^2 of $La_{1.9}Sr_{0.1}Cu_{1-x}Zn_xO_4$

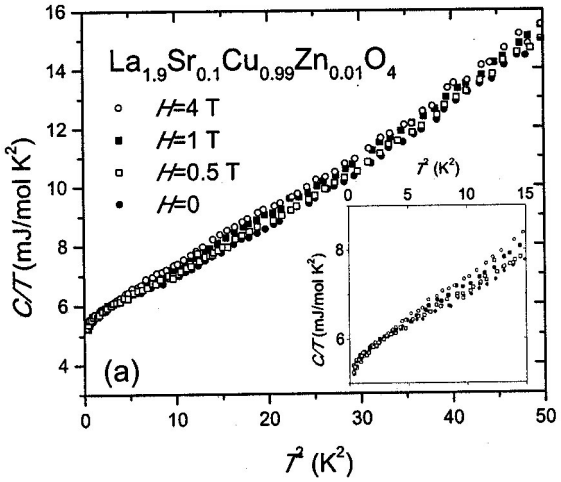

Figure 6: The effects of magnetic fields on $C(T, H)$ of the $x = 0.01$ sample

better fit that $\alpha = 1$, although both values of α qualitatively describe the data. The fitting results for $\alpha = 1/2$ are shown as the solid lines in Fig. 7. This coincides with the case of the thermal metal predicted by the above numerical calculation. If this is the case, it is not clear whether interactions between quasiparticles are the cause of delocalization. But, this assumption is still reasonable since cuprates are know to be systems of strong correlation. $\rho(E) = \rho(0) + A\rho^{\alpha}$ rather than $\rho(E) = \rho(0)$ was clearly observed.

They also measured $C(T, H)$ of $La_{1.9}Sr_{0.1}Cu_{1-x}Ni_xO_4$ and $La_{1.9}Sr_{0.22}Cu_{1-x}Ni_xO_4$. While the results of $La_{1.9}Sr_{0.1}Cu_{1-x}Ni_xO_4$ are the same of $La_{1.9}Sr_{0.1}Cu_{1-x}Zn_xO_4$, C/T of $La_{1.9}Sr_{0.22}Cu_{1-x}Ni_xO_4$ shows no dip. They expected the reason lies in that the underdoped samples are more two-dimensional that the overdoped ones. This can be understood in the light of the quasi-one-dimensional case. These are

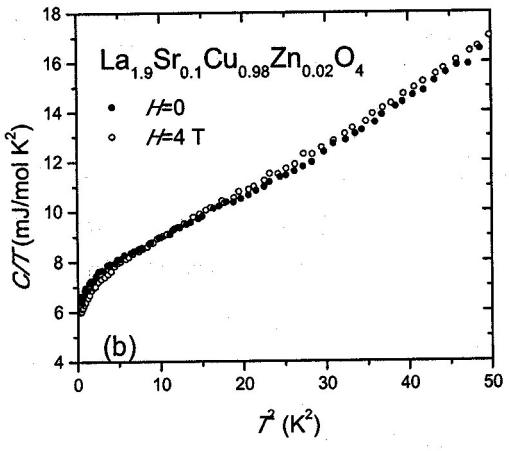

Figure 7: The fitting result of C/T

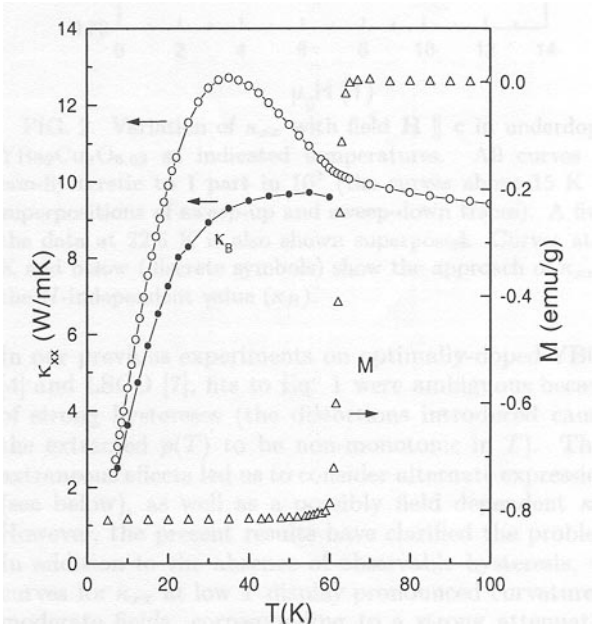

Figure 8: The T dependence of the thermal conductivity $\kappa_{xx}(0)$ in zero field in underdoped YBCO

important results because they provide experimental grounds in accord with the above theory.

For other thermodynamic quantity, thermal conductivity is expected as a very useful probe of the thermal metal-thermal insulator transition. For all superconducting system, the thermal conductivity κ distinguishes the two phases in that the ratio κ/T tends to a finite constant in the thermal metal, and to zero as $T \to 0$ in the thermal insulator. N.P.Ong, K.Krishana, Y.Zhang and Z.A.Xu measured the thermal conductivity, specifically κ_{xx} of superconducting cuprates at various situations. Fig. 8. shows the T dependence of the in-plane thermal conductivity $\kappa_{xx}(0)$ in zero field(open symbols) in underdoped YBCO with $x = 6.63$ and $T_c = 63K$. Here, we can see that the change of κ_{xx} follows the pattern above.

6 Conclusion

The quasiparticle localization transition is well described by this theory and there are also experiment results in accord with predictions. But These experiment are in their first states, so there are not much data to confirm the theory. If interactions which are neglected here, are considered, the situation will be much more complicated, then make both theoretical predictions and experiment difficult. Nonetheless, we have some good candidate for experiment. As a candidate for exhibiting the thermal metal-thermal insulator, the pinned vortex state of a type II s-wave superconductor can be considered. There could exist a critical magnetic field, within the vortex phase at which the low-energy states can permeate through the medium to form extended states. Also a Hall effect in superconducting systems can be considered.

7 References

S. Vishveshwara and M. P. A. Fisher, Phys. Rev. B **64**, 174511(2001)
C. F. Chang, J. Y. Lin, and H. D. Yang, cond-mat/0003022(unpublished)
N. P. Ong, K. Krishana, Y. Zhang, and Z. A. Xu, cond-mat/9904160(unpublished)
P. A. Lee and T. V. Ramakrishnan, Rev, Mod. Phys. **57**, 287(1985)