Coulomb Gas Formulation of Two Dimensional Kosterlitz-Thouless Phase Transition

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Abstract. The absence of conventional long range order in the classical two dimensional *xy* model is introduced. Two main types of excitations of this model, spin wave and vortices, are analyzed in the framework of generalized theory of elasticity. The Hamiltonian is then mapping onto the two dimensional Coulomb gas model and then critical temperature of the Kosterlitz-Thouless transition is located by a renormalization group technique.

1 Introduction

In 1935, Peierls has argued that thermal motion of long-wave length phonons will destroy the long-range order or a two dimensional solid in the sense that the mean square deviation of an atom from its equilibrium position increases logarithmically with the size of the system. In 1968, Mermin proved he absence of long range order of this simple form using rigorous inequalities. Similar arguments can be used to show that there is no spontaneous magnetization in a two-dimensional magnet with spins with more than one degree of freedom and also that the expectation value of the superfluid order parameter in a two dimensional Bose fluid is zero.

On the other hand, it is possible to define a quite different kind of long range order, called *topological* long range order, and there is a phase transition characterized by a sudden change in the response of the system to an external perturbation.

The simplest model of this kind is the two dimensional xy model with O_2 rotational symmetry in a plane. The order parameter that breaks this symmetry can be either a two dimensional vector $\mathbf{s} = s(\cos\phi, \sin\phi)$ or a complex number $\psi = |\psi|e^{i\phi}$, whose respective direction or phase is specified by the angle ϕ. Theoretically, the topological phase transition of this model was first analyzed by Kosterlitz and Thouless[1][2] in 1973. Experimentally, in 1978, Bishop and Reppy[3] studied the superfluid transition of a thin two dimensional helium film absorbed on an oscillating substrate. The observation results on superfluid mass and dissipation supported the Kosterlitz-Thouless picture of the phase transition in a two dimensional superfluid. The jump in the superfluid density at the transition given by Kosterlitz and Thouless is in good agreement with estimates from experiment.

In this term paper, the two dimensional xy model is studied in the framework of the theory of generalized elasticity, and renormalization group analysis is introduced by mapping the system onto a two dimensional Coulomb gas model.

2 The two dimensional *xy* model

The two-dimensional xy model is a system of spins constrained to rotate in the two-dimensional lattice plane. A Note on the name of the model that ***xy*** stands for the two components of each spin

$$\mathbf{s} = (s_x, s_y) = s(\cos\phi, \sin\phi) \tag{1}$$

and ***two dimensional*** indicates the dimensionality of the lattice which, for simplicity, we assume to be a simple square lattice. The Hamiltonian of the system is

$$\mathscr{H} = -J\sum_{\langle ij\rangle} \mathbf{s}_i \cdot \mathbf{s}_j = -J\sum_{\langle ij\rangle} \cos(\phi_i - \phi_j) \tag{2}$$

where the magnitude of the spin s is absorbed into the coupling constant $J > 0$.

2.1 Mermin's proof of the absence of long range order

It has been long suspected before 1960's that one- and two-dimensional *xy* model and Heisenberg model(spin with 3 components instead of 2) with interactions of finite range can be neither ferromagnetic nor antiferromagnetic at non-zero temperature. These conclusions were suggested by the calculation of elementary excitations from the ordered state. Remarkably, in 1967 Mermin[4] was able to prove this plausible result rigorously using the so called *Bogoliubov inequality*:

$$\left\langle |A|^2 \right\rangle \geq \frac{k_B T \, |\langle [C,A^*] \rangle|^2}{\langle [C,[C^*,H]] \rangle} \tag{3}$$

where any physical observable A, C and thermodynamic average with respect to the Hamiltonian H. For two-dimensional *xy* model, the partition function reads

$$e^{-\beta F} = \int_0^{2\pi} \prod_i d\phi_i \times e^{-\beta \mathscr{H}} \tag{4}$$

where symmetry breaking term is added to the Hamiltonian

$$\mathscr{H} = -\sum_{i,j} J_{ij} \cos(\phi_i - \phi_j) - h \sum_i \cos(\phi_i) \tag{5}$$

The canonical variables are ϕ_i and the angular momentum perpendicular to the plane of rotation, say L_i. Take

$$A = \sum_i \sin(\phi_i) e^{-i\mathbf{k}\cdot\mathbf{r}_i}, \quad C = \sum_i L_i e^{-i\mathbf{k}\cdot\mathbf{r}_i} \tag{6}$$

hence

$$[C,H] = -\sum_i e^{-i\mathbf{k}\cdot\mathbf{r}_i} \frac{\partial \mathscr{H}}{\partial \phi_i}, \quad [C,[C^*,H]] = \sum_{ij} e^{i\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_j)} \frac{\partial^2 \mathscr{H}}{\partial \phi_i \partial \phi_j} \tag{7}$$

so that

$$\begin{aligned}
\langle [C,[C^*,H]] \rangle &= \sum_{ij} e^{i\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_j)} \left\langle \frac{\partial^2 \mathscr{H}}{\partial \phi_i \partial \phi_j} \right\rangle \\
&= 2\sum_{ij} J_{ij} \left(1 - e^{i\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_j)}\right) \left\langle \cos(\phi_i - \phi_j) \right\rangle + h \sum_i \langle \cos\phi_i \rangle \\
&= 2\sum_i \sum_{\mathbf{r}_{ij}} J_{ij} \Big(1 - \cos(\mathbf{k}\cdot\mathbf{r}_{ij})\Big) \left\langle \cos(\phi_i - \phi_j) \right\rangle + h \sum_i \langle \cos\phi_i \rangle \\
&\leq \sum_i \sum_{\mathbf{r}_{ij}} |J_{ij}| \times 4 \sin^2\left(\frac{\mathbf{k}\cdot\mathbf{r}_{ij}}{2}\right) + N|h||m| \\
&\leq \sum_i \sum_{\mathbf{r}_{ij}} k^2 |J_{ij}| \, r_{ij}^2 + N|h||m| \\
&= N \left\{ k^2 \sum_{\mathbf{r}} |J(r)| \, r^2 + |h||m| \right\}
\end{aligned}$$

Finite range of interaction ensures that $\sum_{\mathbf{r}} |J(r)| r^2 < \infty$. On the other hand,

$$\begin{aligned} [A^*, C] &= \sum_{ij} \left[\sin\phi_i, L_j\right] e^{i\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_j)} \\ &= \sum_{ij} \delta_{ij} \cos\phi_i e^{i\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_j)} \\ &= \sum_i \cos\phi_i \end{aligned}$$

so that $\langle [A^*, C] \rangle = Nm$, where m is the magnetization per site $m = N^{-1} \sum_i \langle \cos\phi_i \rangle$; and

$$|A|^2 = \sum_{ij} \sin\phi_i \sin\phi_j e^{i\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_j)} \tag{8}$$

Substitute the above results into Bogoliubov inequality

$$\sum_{ij} \left\langle \sin\phi_i \sin\phi_j \right\rangle e^{i\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_j)} \geq \frac{N k_B T m^2}{k^2 \sum_{\mathbf{r}} |J(r)| r^2 + |h||m|} \tag{9}$$

sum over $\mathbf{k}$ we get

$$N \sum_i \left\langle \sin^2\phi_i \right\rangle \geq \sum_{\mathbf{k}} \frac{N k_B T m^2}{k^2 \sum_{\mathbf{r}} |J(r)| r^2 + |h||m|} \tag{10}$$

or

$$\frac{k_B T m^2}{N} \sum_{\mathbf{k}} \frac{1}{k^2 \sum_{\mathbf{r}} |J(r)| r^2 + |h||m|} \leq \frac{1}{N} \sum_i \left\langle \sin^2\phi_i \right\rangle \leq 1 \tag{11}$$

In the thermodynamics limit this becomes ($n \equiv N/V$)

$$\frac{k_B T m^2}{n} \int \frac{d^D\mathbf{k}}{(2\pi)^D} \frac{1}{k^2 \sum_{\mathbf{r}} |J(r)| r^2 + |h||m|} \leq 1 \tag{12}$$

This integral has infrared divergence in one and two dimension, which requires that m vanishes as h goes to zero for any non-zero temperature. This completes the proof.

2.2 Spin wave excitations: fluctuation destruction of long range order

Mermin's proof is rigorous but abstract. In this section, the physical interpretation of the absence of spontaneous magnetization is shown to be the instability of the ground state with long range order against low-energy spin-wave excitations. We have to make approximation to exchange for the desirable intuitive physical picture. This approximation is made upon the Hamiltonian by noticing that only slowly varying configurations, that is, those with adjacent angles nearly equal, will give any significant contribution to the partition function so that the Hamiltonian may be expanded up to quadratic terms in the angles.

$$\begin{aligned} \mathscr{H} &= -J \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j) = E_0 + \frac{1}{2} J \sum_{\langle i,j \rangle} \left(\phi_i - \phi_j\right)^2 \\ &\sim J \int d^2\mathbf{r} \left(\nabla\phi(\mathbf{r})\right)^2 \\ &= \frac{J}{V} \sum_{\mathbf{q}} \mathbf{q}^2 \left|\phi_{\mathbf{q}}\right|^2 \end{aligned}$$

where the continuum limit and Fourier transform are taken in the last two equalities. This is called the *elastic approximation* of the Hamiltonian $\mathscr{H}_{el}$. Quote the classical equipartition theorem, we have

$$\left\langle |\phi_{\mathbf{q}}|^2 \right\rangle = \frac{k_B T V}{2J\mathbf{q}^2} \tag{13}$$

The magnetization

$$\langle \cos\phi \rangle = \frac{1}{Z_{el}} \int \mathscr{D}\phi e^{-\beta \mathscr{H}_{el}} \cos(\phi) \approx e^{-\frac{1}{2}\langle \phi^2(\mathbf{r}) \rangle}$$

is evaluated approximately by the fluctuation of the angles, say

$$\begin{aligned} \left\langle \phi^2(\mathbf{r}) \right\rangle &= \sum_{\mathbf{q},\mathbf{q}'} \left\langle \phi_{\mathbf{q}} \phi_{\mathbf{q}'} \right\rangle e^{i(\mathbf{q}+\mathbf{q}')\cdot \mathbf{r}} \\ &= \sum_{\mathbf{q}} \left\langle |\phi_{\mathbf{q}}|^2 \right\rangle \\ &= \frac{k_B T V}{2J} \sum_{\mathbf{q}} \frac{1}{\mathbf{q}^2} \\ &= \frac{k_B T V}{2J} \int \frac{d^2\mathbf{q}}{\mathbf{q}^2} \end{aligned}$$

in which the integral again has infrared divergence in two dimension and there is no long range order. This is the phenomena of *fluctuation destruction of long range order*. Physically, a very long wavelength rotation of the direction of $\langle \mathbf{s} \rangle$ costs very little energy because of the continuous degree of freedom in contrast to the two-dimensional Ising model. Thermal excitation of these long wavelength modes changes the direction of the magnetization in space and time and can greatly reduce the total magnetization from the value when all spins are aligned.

2.3 Vortex type excitations

In 1960's there is some numerical evidence that 2D *xy* model has a phase transition, but Mermin's proof indicates that it cannot be the usual type with spontaneous magnetization below some critical temperature. The problem is that spin wave is not the only relevant kind of excitations, but also the vortex type excitations should be taken into account. Write the Hamiltonian in elastic approximation

$$\mathscr{H} = J \int d^2\mathbf{r} \left(\nabla \phi(\mathbf{r})\right)^2 \tag{14}$$

This energy functional is minimized by the following variational calculation neglecting boundary contributions

$$\frac{\delta \mathscr{H}}{\delta \phi(\mathbf{r})} = -2J\nabla^2 \phi(\mathbf{r}) = 0 \quad \Rightarrow \quad \nabla^2 \phi(\mathbf{r}) = 0 \tag{15}$$

The vortex configuration of Fig. 1 is a possible kind of solution, for example

$$\phi(\mathbf{r}) = q\varphi(\mathbf{r}) + \phi_0 \tag{16}$$

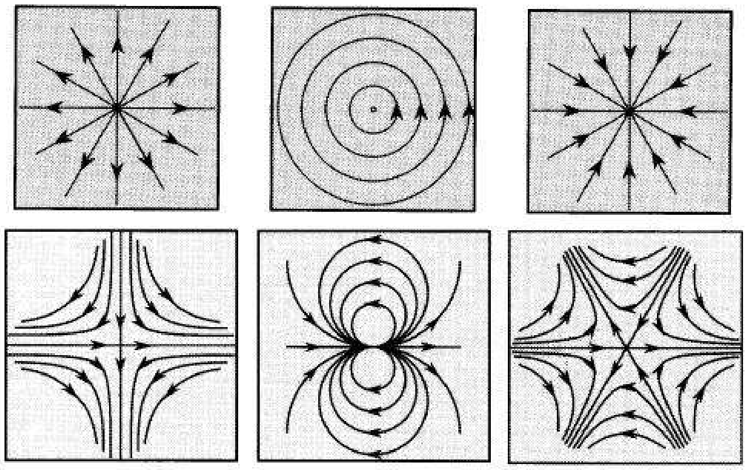

Figure 1. Single vortex configurations with $k = 1$ for the upper three configurations and $k = -1, 2, -2$ for the lower three configurations

where φ is the polar angle corresponding to $\mathbf{r}$ in the plane polar coordinate system and the integer q is called the *vorticity* or *winding number* of the vortex which characterizes the strength of the vortex. Then

$$\nabla\phi(\mathbf{r}) = \frac{q}{r}\mathbf{e}_\varphi \tag{17}$$

and the elastic energy of this vortex configuration is easily calculated

$$E_{el} = J\int_a^R\int_0^{2\pi} rdrd\varphi\frac{q^2}{r^2} = 2\pi q^2 J\log\left(\frac{R}{a}\right) \tag{18}$$

where a is the lattice spacing or core radius of the vortex, and R is the linear dimension of the system. An alternative calculation of elastic energy of a single vortex configuration is more instructive and is more easily generalized to the calculation of interaction energy between two vortices. Recall that $\nabla^2\phi(\mathbf{r}) = 0$, and carry out the boundary integration along the cut connecting the vortex and the boundary of the system as in Fig. 2(a)

$$\begin{aligned}
E_{el} &= J\int_\Omega d^2\mathbf{r}\,(\nabla\phi(\mathbf{r}))^2 \\
&= J\int_{\partial\Omega}\phi(\mathbf{r})\nabla\phi(\mathbf{r})\cdot d\mathbf{r} - J\int_\Omega \phi(\mathbf{r})\nabla^2\phi(\mathbf{r})d^2\mathbf{r} \\
&= J\int_{\text{Inner}}\phi(\mathbf{r})\nabla\phi(\mathbf{r})\cdot d\mathbf{r} + J\int_{\text{Outer}}\phi(\mathbf{r})\nabla\phi(\mathbf{r})\cdot d\mathbf{r} \\
&\quad + J\int_{\text{Cut}+}\phi(\mathbf{r})\nabla\phi(\mathbf{r})\cdot d\mathbf{r} + J\int_{\text{Cut}-}\phi(\mathbf{r})\nabla\phi(\mathbf{r})\cdot d\mathbf{r} \\
&= J\left(\phi^+ - \phi^-\right)\int_a^R \nabla\phi(\mathbf{r})\cdot d\mathbf{r} \\
&= 2\pi q^2 J\log\left(\frac{R}{a}\right)
\end{aligned}$$

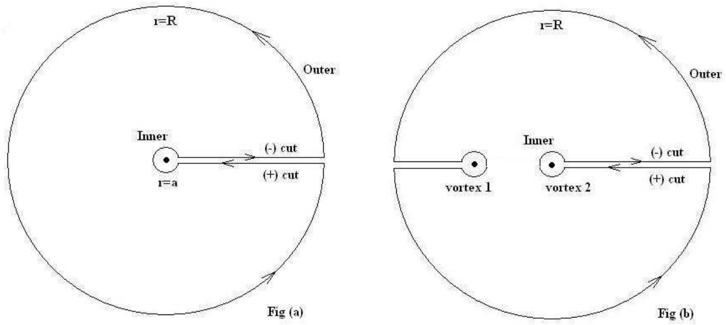

Figure 2. Energy of a single vortex in (a) and interaction energy between two vortices in (b)

The interaction energy between two vortices can be calculated in a similar manner by introducing a cut for each vortex as in Fig. 2(b)

$$\begin{aligned} E_{el} &= J\int d^2\mathbf{r}\,(\nabla\phi_1(\mathbf{r})+\nabla\phi_2(\mathbf{r}))^2 \\ &= E_1+E_2+2J\int d^2\mathbf{r}\nabla\phi_1(\mathbf{r})\cdot\nabla\phi_2(\mathbf{r}) \\ &= E_1+E_2+2J\int \phi_1(\mathbf{r})\nabla\phi_2(\mathbf{r})\cdot d\mathbf{r} \\ &= E_1+E_2+2J\left(\phi_1^+-\phi_1^-\right)\int_r^R \nabla\phi_2(\mathbf{r})\cdot d\mathbf{r} \\ &= E_1+E_2+4\pi q_1 q_2 J\log\left(\frac{R}{r}\right) \\ &= 2\pi\left(q_1+q_2\right)^2 J\log\left(\frac{R}{a}\right)-4\pi q_1 q_2 J\log\left(\frac{r}{a}\right) \end{aligned}$$

A more explicit derivation of the above formula for the interaction energy of two vortices in a cylinder can be found in ShengQuan Zhou's solution to the Homework Sheet 1 for Phys.569 *Emergent state of matter*. This formula can be easily generalized to many-vortices configuration

$$E_{\text{vortices}} = -2\pi J\sum_{i\neq j} q_i q_j \log\left|\frac{\mathbf{r}_i-\mathbf{r}_j}{a}\right| + 2\pi J\left(\sum_i q_i\right)^2 \log\left(\frac{R}{a}\right), \quad \text{for}\left|\mathbf{r}_i-\mathbf{r}_j\right|>a$$

There are two things to note about this equation. First, the term with $\log R$ diverges in thermodynamic limit and thus, in an infinite sample, there is an infinite energy cost associated with deviations of the total vorticity from zero. This divergence is eliminated if the total vorticity is zero, i.e. $\sum_i q_i = 0$. Thus, configurations containing vortices whose winding number satisfy $\sum_i q_i = 0$ have energies that do not diverge with the sample size and only these states will be thermally excited for $T > 0$. Secondly, the core radius a remains unspecified. It is a variational parameter which will adjust to minimize

$$E_{\text{vortex}} = E_{\text{el}} + E_{\text{core}} \tag{19}$$

The core energy at the optimal value of a grows quadratically with the vorticity q

$$E_{\text{core}} = \pi^2 J q^2 \tag{20}$$

2.4 Mechanism of Kosterlitz-Thouless transition: energy-entropy argument

Now we are in a good position to explain the mechanism of the phase transition named after Kosterlitz and Thouless. Besides the usual spin-wave excitations which are responsible for ensuring that the mean magnetization is zero at all finite temperatures, there are other equilibrium configurations which are not taken into account in the usual treatments before 1960's. These are vortex configurations. The energy of an isolated vortex configuration is given by

$$E \approx 2\pi J \log\left(\frac{R}{a}\right) \tag{21}$$

Since there are $(R/a)^2$ possible positions for the vortex, its entropy is

$$S = 2k_B \log\left(\frac{R}{a}\right) \tag{22}$$

The free energy is

$$F = 2\left(\pi J - k_B T\right)\log\left(\frac{R}{a}\right) \tag{23}$$

At sufficiently low temperatures, the energy dominates so that it is unfavorable for isolated vortices to occur, while at high temperatures the entropy terms takes over and isolated vortices occur. The critical temperature is approximately

$$k_B T_c = \pi J \tag{24}$$

This estimation of critical temperature can be improved by renormalization group analysis outlined in the following sections.

3 Coulomb gas formulation of Kosterlitz-Thouless transition

As discussed above, we must take into account the vortex configurations as well as the spin-wave excitations which are responsible for destroying the conventional long-range order. Note that the *vorticity* q of a given region may be defined by

$$\oint d\phi(\mathbf{r}) = 2\pi q \tag{25}$$

where the integral is taken around the boundary of the region. This suggests that we define

$$\phi(\mathbf{r}) = \psi(\mathbf{r}) + \bar{\phi}(\mathbf{r}) \tag{26}$$

where $\bar{\phi}(\mathbf{r})$ defines the angular distribution of the configuration of the variational local minimum of $\mathscr{H}$

$$\frac{\delta\mathscr{H}}{\delta\bar{\phi}(\mathbf{r})} = 0 \tag{27}$$

The absolution minimum is give by the uniform configuration $\bar{\phi}(\mathbf{r}) = \text{const}$ and the other local minima are vortex configurations with $q \neq 0$; and $\psi(\mathbf{r})$ defines the deviations from this local minimum or spin waves. The energy of this configuration is therefore

$$\mathscr{H} = J\int d^2\mathbf{r}\left(\nabla\bar{\phi}(\mathbf{r})\right)^2 + J\int d^2\mathbf{r}\,(\nabla\psi(\mathbf{r}))^2 \tag{28}$$

where

$$\int d\mathbf{r}\cdot\nabla\bar{\phi}(\mathbf{r}) = 2\pi q, \quad \int d\mathbf{r}\cdot\nabla\psi(\mathbf{r}) = 0 \tag{29}$$

The cross term vanishes because of the above conditions. The above conditions also defines the vorticity density $\rho(\mathbf{r}) = \sum_i q_i\delta(\mathbf{r}-\mathbf{r}_i)$. Since

$$2\pi\int\rho(\mathbf{r})d^2\mathbf{r} = 2\pi q = \oint d\phi(\mathbf{r}) = \oint d\mathbf{r}\cdot\nabla\phi(\mathbf{r}) = \int d^2\mathbf{r}\nabla^2\phi(\mathbf{r}) \tag{30}$$

we have the Poisson equation

$$\nabla^2\bar{\phi}(\mathbf{r}) = 2\pi\rho(\mathbf{r}) \tag{31}$$

In terms of ρ, the energy of the system in a given configuration is

$$\mathscr{H} \;=\; J\int d^2\mathbf{r}\,(\nabla\psi(\mathbf{r}))^2 - 2\pi J\int\int d^2\mathbf{r}d^2\mathbf{r}'\rho(\mathbf{r})\rho(\mathbf{r}')\log\left|\frac{\mathbf{r}-\mathbf{r}'}{a}\right| + E_c\sum_i q_i^2$$

The last term in this expression is the core energy where $E_c = \pi^2 J$. Note that the integral over positions contains a short distance curoff preventing two vortices from occupying the same position in space. Since there is a minimum distance between vortices, the lattice representation is often convenient. Also, the requirement that the total vorticity of the system vanishes, i.e. $\sum_i q_i = 0$ has been taken into account. For simplicity we make further assumption that only vortices with unit strength are present since these will be much more favorable than vortices with $|q| > 1$. The partition function Z may now be written as

$$Z = \mathrm{Tr}\exp(-\beta\mathscr{H}_{2n}) \tag{32}$$

where

$$\mathscr{H}_{2n} = -\sum_{i\neq j} q_i q_j \log\left|\frac{\mathbf{r}_i-\mathbf{r}_j}{a}\right| + 2nE_c \tag{33}$$

and the trace

$$\mathrm{Tr} \equiv \int\mathscr{D}\psi(\mathbf{r})\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\int_{D_{2n}}\frac{d^2\mathbf{r}_{2n}}{a^2}\cdots\int_{D_1}\frac{d^2\mathbf{r}_1}{a^2} \tag{34}$$

where $\beta = 1/k_BT$. The functional integral over the angle $\psi(\mathbf{r})$ is carried out for $-\infty < \psi(\mathbf{r}) < \infty$, although, strictly speaking $|\psi(\mathbf{r})| < \pi$. Since the Hamiltonian is quadratic in $\psi(\mathbf{r})$, such a change should have a negligible effect. We must then further integrate over all possible positions of the vortices which are allowed to go over the whole plane subject only to the restriction $|\mathbf{r}_i - \mathbf{r}_j| \geq a$. Finally we sum over all possible numbers of vortices and include a statistical factor $(n!)^{-2}$ to allow for the fact that there are two sets of n vortices of equal but opposite vorticity.

3.1 Mapping 2D *xy* model to Coulumb gas

Since in our approximation, the spin wave and vortices do not interact with each other, the problem is now reduced to that of a model system with the vortices playing the roles of the charged particles in two dimension. Apart from the core contribution, the vortex part of $\mathscr{H}$ is identical to the Hamiltonian of a two-dimensional Coulomb gas with point charges $q_i = \pm\sqrt{2\pi J}$

$$\mathscr{H}_C \;=\; -\sum_{i\neq j} q_i q_j \log\left|\frac{\mathbf{r}_i-\mathbf{r}_j}{a}\right|$$

and the charge neutrality constraint $\sum_i q_i = 0$. However, if we form the grand canonical partition function for such system,

$$Z = \sum_n \frac{e^{-2n\beta\mu}}{(n!)^2} \int_{D_{2n}} \frac{d^2\mathbf{r}_{2n}}{a^2} \cdots \int_{D_1} \frac{d^2\mathbf{r}_1}{a^2} \exp\left(-\beta \mathscr{H}_C\right) \tag{35}$$

the core contribution of the energy plays the role of chemical potential, where $\mu = E_c$ and with the restriction that the system is dilute, so that $\mu/k_B T$ is large, then we have precisely the same system as the classical plasma.

3.2 Renormalization group analysis of Coulomb gas: basic idea

The continuum Coulomb gas model is renormalized according to a scheme developed by Kosterlitz(1974). The basic idea is to rescale the lattice spacing or the minimum particle separation. The parameter a sets the microscopic length-scales in the problem in three places, in the argument of the Coulomb potential, in the domain of integration by the constraint that particles are separated by a distance of at least a, and in the area element of the same integral. The partition sum of this Coulomb gas model is reformulated in terms of new effective particles infinitesimally large than the original, with a hard-core diameter $a+da$. The partition function governing the new particles is expected to be of the same functional form as the original, so that the rescaling procedure can be iterated. The first order variation of the thermodynamic parameters gives the RG recursion relations.

The integration domain of Eq.(35) can be divided into two parts, one in which all particles are separated by more than $a+da$, and another one in which at least two particles are located at a mutual distance between a and $a+da$. The configurations in the former sub-domain are kept as states of the rescaled particles, and the integration over the latter is carried out in part, to contribute to effective configurations of the new particles. To first order in da it is sufficient to consider only configurations with at most one pair of particles, say i and j, with relative position in the annulus $a \le |\mathbf{r}_i - \mathbf{r}_j| \le a+da$. This pair is adopted in the rescaled system as a single particle with charge $q = q_i + q_j$. When this total charge is zero the effective particle would be neutral, which would imply an undesirable generalization of the original Coulomb gas. Therefore, these neutral complexes are considered as unoccupied space, and their position is integrated out. In summary, the thermodynamic parameters are renormalized by three different processes: (1)the change of the scale parameter in the Coulomb potential and in the integration; (2)the fusion of two particles forming a charged complex; and (3)the annihilation of two oppositely-charged particles. To linear order in da these three effects can be considered independently. In the following, we shall skip the derivation of recursion relations. However, the main contribution to Z will come from pairing particles with opposite charge so that may restrict the sum over i and j to be over particles with $q_i = -q_j$.

3.3 Scaling equations

On scaling the lattice spacing a to $a+da$, we recover the partition function with unchanged functional form but with scaled interaction parameters, denoted by tilde, given by the following equations

$$\begin{aligned} \beta\tilde{q}^2 &= \beta q^2\left[1-4\pi^2\left(\beta q^2\right)z^2\frac{da}{a}\right] \\ \tilde{z} &= z\left[1-\left(\beta q^2-2\right)\frac{da}{a}\right] \end{aligned}$$

where $\beta = 1/k_BT$, $q^2 = 2\pi J$, and the fugacity $z = e^{-\beta\mu}$. There is one fixed point solution to these equations at

$$\begin{aligned} \beta q^2 - 2 &= 0 \\ z &= 0 \end{aligned}$$

Define $x = \beta q^2 - 2$ and $y = 4\pi z$. Write these equations in differential form and linearizing about this fixed point,

$$\begin{aligned} dx &= -y^2\frac{da}{a} \\ dy^2 &= -2xy^2\frac{da}{a} \end{aligned}$$

These equations are easily integrated to obtain

$$x^2 - y^2 = \text{const} \tag{36}$$

Thus an improved estimate of the critical temperature T_c is given by $x^2 - y^2 = 0$

$$\frac{\pi J}{k_BT_c} - 1 = 2\pi\exp\left(-\frac{\pi^2 J}{k_BT_c}\right) \tag{37}$$

Solving this equation numerically gives

$$k_BT_c \cong 0.86\pi J \tag{38}$$

3.4 Effect of Kosterlitz-Thouless transition: susceptibility

The existence of a fixed point at nonzero temperature signifies a phase transition, which is called Kosterlitz-Thouless transition. Below the critical temperature T_c, the vortices will be bound in pairs of zero total vorticity, while above T_c they are free to move to the surface under the influence of an arbitrary weak applied magnetic field, thereby causing a sudden change in the form of the response to the applied field.

One way of seeing this change is to consider the spin-spin correlation function $\left\langle \mathbf{s}_i \cdot \mathbf{s}_j \right\rangle$ since the static sum rule or fluctuation-dissipation theorem would then give the information of susceptibility. Taking the vortices into account, this quantity read

$$\left\langle \mathbf{s}_i \cdot \mathbf{s}_j \right\rangle = \left\langle \exp\left[i(\psi_i - \psi_j)\right]\right\rangle \left\langle \exp\left[i(\bar{\phi}_i - \bar{\phi}_j)\right]\right\rangle = \Gamma_v\left(\mathbf{r}_i - \mathbf{r}_j\right)\Gamma_s\left(\mathbf{r}_i - \mathbf{r}_j\right) \tag{39}$$

Since the vortices and spin waves do not interact in our approximation, the two averages are taken independently. At the critical temperature $T = T_c$,

$$\Gamma_v(r) \sim r^{-\frac{1}{4}} \tag{40}$$

The critical exponents $\eta = 1/4$ comes from a conspiracy between the spin wave and vortex contribution. The spin wave alone gives $\eta = k_B T_c / 4\pi J$. For $T < T_c$, $\Gamma(r)$ falls of somewhat more slowly than $r^{-1/4}$, but its precise behavior is not of interest. However, we have, from the fluctuation-dissipation theorem, the susceptibility

$$\chi \sim \int d^2\mathbf{r}\Gamma(r) = \infty, \quad T \leq T_c \tag{41}$$

Above T_c, there are free mobile vortices interacting via a Coulomb potential. In effect, we must calculate $\Gamma_v(r)$ for high temperatures

$$\Gamma(r) \sim r^{-\frac{1}{4}} f\left(\frac{r}{\xi}\right) \tag{42}$$

where $f(r/\xi)$ falls off exponentially for large r and the susceptibility

$$\chi \sim \int_{\xi}^{\infty} d^2\mathbf{r} r^{-\frac{1}{4}} f\left(\frac{r}{\xi}\right) \sim \xi^{\frac{7}{4}}, \quad T > T_c \tag{43}$$

4 Conclusion: topological long range order

As proved by Mermin, systems with a broken *continuous* symmetry in two spatial dimensions do not have long range order. Order parameter correlation functions at low temperature in systems with *xy*-type symmetry do, however, dei off algebraically rather than exponentially, as they would in completely disordered high temperature phases. In two dimensional *xy* model, the Kosterlitz-Thouless transition is just such a transition from quasi-long-range order to disorder.

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