Continuum Model of Avalanches in Granular Media

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Abstract

A continuum description of avalanches in granular systems is presented. The model is based on hydrodynamic equations coupled with an order parameter determined by a free-energy-like potential. The model successfully describes the transition between the static and the fluidized phases. This formalism is applied to avalanches on an inclined plane configuration. The theoretical predictions agree with the experimental results.

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1 Introduction

Granular materials are large conglomeration of discrete macroscopic particles with no cohesive forces between them. In spite of this simplicity, they can exhibit very rich and surprising phenomena that differ from those seen in solids, liquids or gases [1]. Example of these behaviors are segregation under vertical vibration [2], localized excitations [3] and avalanches.

Grains are macroscopic, so thermal energy is negligible compared to the gravitational energy of the particles ($k_BT/mgd \approx 10^{-10}$, where m and d are the mass and diameter of the particles, and g is the gravitational constant). Consequently, in most cases ergodicity does not apply. The number of micro states corresponding to a macro state is irrelevant. Additionally, the interactions between grains are highly dissipative due to the static friction and the inelastic collisions experienced by the particles. These two characteristics make the ordinary statistical mechanics fail in describing granular materials.

One of the most interesting phenomena pertinent to granular systems is *avalanches*. During an avalanche, a surface layer of the material experiences a phase transition from a static state to a granular flow. Figures 1.a and 1.b show avalanche experiments done on an inclined plane.

The description of avalanches is still a matter of debate. The two more successful approaches to describe avalanches are: (a) large-scale molecular dynamics simulations, and (b) continuum theories. The first continuos models of avalanches were studied by Bouchaud, Cates, Ravi Prakash and Edwards (BCRE) [4], and subsequently by Boutreux, Raphaël and de Gennes [5]. In their models, the granular system is spatially separated into two phases, static and flow. However, in many important cases there is not a clear distinction between those two states.

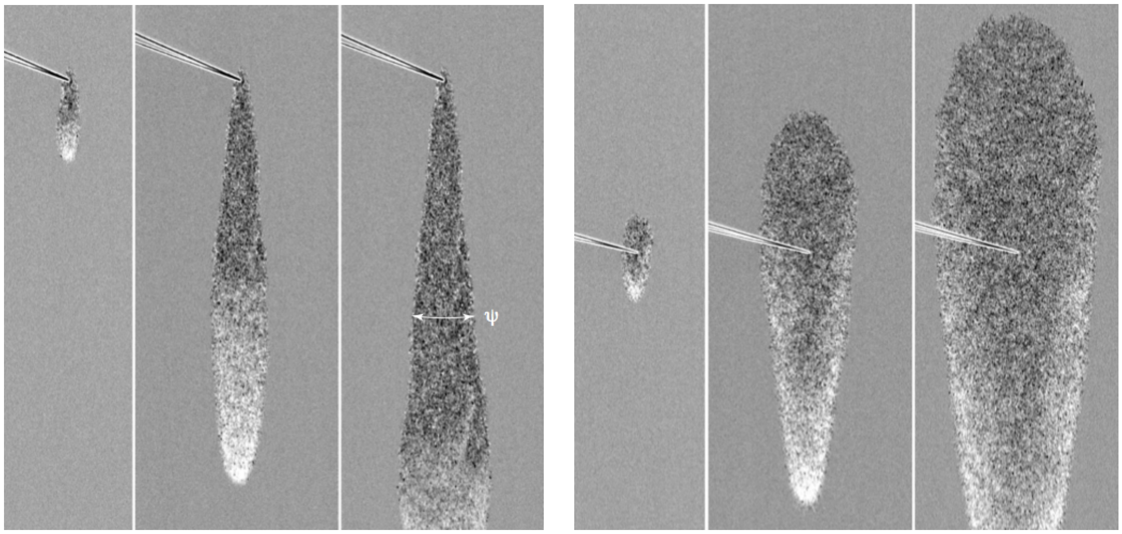
In this essay, the Aranson-Tsimring [6, 7, 8] model of avalanches will be presented.

2 Model of granular flows

In the Aranson-Tsimring approach, the model is based on the Navier-Stokes equations

$$\rho_0 D_t v_i = \partial_j \sigma_{ij} + \rho_0 g_i, \quad i, j = x, y, z \tag{1}$$

where $\mathbf{v}$ is the velocity of the material, ρ_0 is the density (without loss of generality, $\rho_0 = 1$), σ_{ij} are the components of the stress tensor, $\mathbf{g}$ is the gravitational acceleration, and $D_t = \partial_t + v_i \partial_i$ is the material derivative. Also, a dense flow will be



(a) (b)

Figure 1: Avalanches triggered by a needle. Initially, an uniform layer of granular particles rest on an inclined plane. Depending on the tilt angle, two different behaviors are seen. (a) Triangular avalanche. (b)Up-hill avalanche. [9]

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assume, so the relative density fluctuations are small and the fluid will be considered incompressible, $\nabla \times \mathbf{v} = 0$.

The main assumption in this theory is that there are two components to the granular flow. One involving grains experiencing plastic motion, and another term accounting for grains that maintain static contacts with their neighbors. In this way, the stress tensor is written as the sum of a hydrodynamic part proportional to the flow strain rate e_{ij}, and a strain-independent part σ^s_{ij}

$$\sigma_{ij} \equiv \eta \, e_{ij} + \sigma^s_{ij} \tag{2}$$

where η is the viscosity coefficient.

It is assumed that the diagonal elements of the stress tensor σ^s_{ii} are equal to the diagonal components of the static stress tensor σ^0_{ii} for the immobile grain configuration, and the shear stresses are proportional to the stress tensor σ^0_{ij}, where the constant of proportionality is a function of a position-dependent order parameter ρ.

For simplicity, $\sigma^s_{ij} \equiv \rho\, \sigma^0_{ij}$, for $i \neq j$. In this way,

$$\sigma_{ij} = \eta\, (\partial_j v_i + \partial_i v_j) + \sigma^s_{ij} \tag{3}$$

where

$$\sigma^s_{ij} = \begin{cases} \sigma^0_{ii} & i = j \\ \rho\, \sigma^0_{ij} & i \neq j \end{cases} \tag{4}$$

The static or *solid* state is identified as $\rho = 1$. In this way, the stress tensor σ_{ij} takes its static value, $\sigma_{ij} = \sigma^0_{ij} \quad \forall i, j$. In a fluidized or *fluid* state, $\rho = 0$ and the stress tensor is reduced to the case of an ordinary fluid.

Because of the strong dissipation in a dense granular flow, the equation for the order parameter ρ is determined by a pure dissipative dynamical equation $D_t\rho = -\delta\mathcal{F}/\delta\rho$, where $\mathcal{F}$ is a free energy density functional, $\mathcal{F} = \frac{1}{2}D|\nabla\rho|^2 + F(\rho)$. The constant D is the diffusion coefficient (without loss of generality, $D = 1$), and $F(\rho)$ is a potential energy density.

$$D_t\rho = \nabla^2\rho - \partial_\rho F(\rho) \tag{5}$$

To account for the bistability near the solid-fluid transition, $F(\rho)$ is assumed to have local minima at $\rho = 1$ and $\rho = 0$. For simplicity, $F(\rho)$ is taken as a quartic form

$$F(\rho) = \int^{\rho} \rho'(\rho' - 1)(\rho' - \delta)d\rho' \tag{6}$$

Therefore,

$$D_t\rho = \nabla^2\rho + \rho(1 - \rho)(\rho - \delta) \tag{7}$$

The position-dependent parameter δ controls the stability of the two phases. For $0 < \delta < 1$, Eq. (7) has two stable uniform solutions at $\rho = 0$ and $\rho = 1$, corresponding to liquid and solid states, and one unstable solution at $\rho = \delta$ (see figure 2). For $\delta < 1/2$, the solid phase is more favorable. For $\delta > 1/2$, the liquid phase is more favorable.

The parameter δ is taken to be a function of the stress tensor σ_{ij}, which depend on constitutive relations. One of the simplest functions for δ is $\delta = (\phi - \phi_0)/(\phi_1 - \phi_0)$, where $\phi \equiv \max\limits_{i,j} |\sigma_{ij}/\sigma_{jj}|$, and ϕ_0, ϕ_1 are critical parameters that account for hysteresis in the system. If $\phi < \phi_0 \Rightarrow \delta < 0$ and the liquid state is unstable. If $\phi > \phi_1 \Rightarrow \delta > 1$ and the solid state is unstable. If $\phi_0 < \phi < \phi_1$, both states coexist.

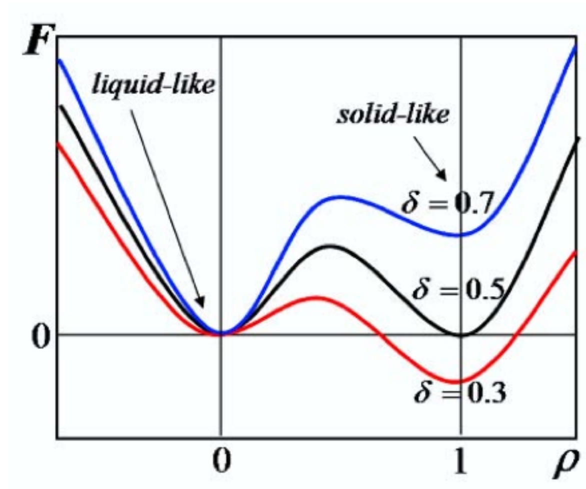

Figure 2: $F(\rho)$ for three different values of δ [8]

3 Inclined plane configuration

A particular configuration for granular flows consists in an initially uniform flat layer of thickness h of dry cohesionless grains on a plane tilted at an angle φ. A Cartesian coordinate system is placed as shown in Fig. (3), where the z axis is perpendicular to the plane and the x axis goes downhill. The bottom of the layer is at $z = -h$ and the top of it is at $z = 0$. The surface of the plain will be considered sticky, so a non-slip condition will be imposed on the grains at the bottom of the granular layer, i.e., $\rho = 1$ at $z = -h$. At the free surface the boundary condition is less obvious. For simplicity, a non-flux boundary condition for the order parameter will be imposed, $\partial_z \rho = 0$.

3.1 Uniform stationary solutions

The uniform stationary solution for the order parameter is simply $\rho = 0, 1$ (Eq. 7), where the stability of each phase is determined by the configuration of the system. The uniform stationary solution for the stress tensor σ_{ij} follows from Eq. (1),

$$\partial_z \sigma_{zz} + \partial_x \sigma_{zx} - g \cos\varphi = 0 \tag{8}$$

$$\partial_z \sigma_{xz} + \partial_x \sigma_{xx} + g \sin\varphi = 0 \tag{9}$$

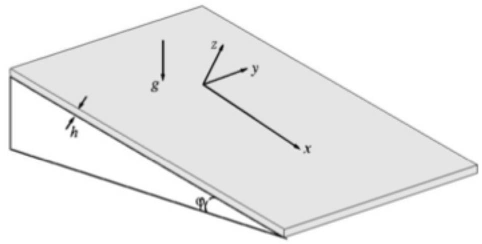

Figure 3: Inclined plane configuration.

In the absence of lateral stresses, i.e., $\sigma^0_{yy} = \sigma^0_{yx} = \sigma^0_{yz} = 0$, the solution is

$$\sigma_{zz} = g\cos\varphi z, \quad \sigma_{xz} = -g\sin\varphi z, \quad \sigma_{xx,x} = 0 \tag{10}$$

Therefore, there is a simple relation between shear and normal stresses, $\sigma_{xz} = -\sigma_{zz}\tan\varphi$, independent of the flow profile. In the case of static equilibrium, $\sigma^0_{xz} = -\sigma^0_{zz}\tan\varphi$. Since by assumption $\sigma_{zz} = \sigma^0_{zz}$, then $\sigma_{xz} = \sigma^0_{xz}$.

In this configuration, the most unstable direction goes along the inclined plane. Hence,

$$\phi = \max_{i,j}|\sigma_{ij}/\sigma_{jj}| = |\sigma^0_{xz}/\sigma^0_{zz}| = \tan\varphi \tag{11}$$

Consequently, the stability of this solution is totally determined by the tilt angle of the plane φ. Explicitly,

$$\delta = \frac{\phi - \phi_0}{\phi_1 - \phi_0} \tag{12}$$

where $\phi_i \equiv \tan\varphi_i$ for $i = 1, 2$, and φ_0, φ_1 are critical tilt angles.

3.2 Non-uniform solutions

For the case of non-uniform flows, solutions with small deviations from the uniform case will be considered, i.e., $|\partial_x h(x, y, z)| \ll \tan\varphi$. Additionally, a further approximation will be made. Because of the non-slip condition at the bottom of the granular layer, for shallow layers the flow velocity is small [7], so the convective flux can be neglected, i.e., $D_t \approx \partial_t$,

$$\partial_t\rho = \nabla^2\rho + \rho(1-\rho)(\rho - \delta) \tag{13}$$

3.2.1 Stationary case

For h greater than a critical thickness h_s, there is a non-trivial stationary solution. The critical thickness h_s can be derived in the following way.

Integrating Eq. (6) gives,

$$F(\rho) = \frac{\rho^4}{4} - \frac{(\delta + 1)}{3}\rho^3 + \frac{\delta}{2}\rho^2 + const. \tag{14}$$

In a stationary regime, $\partial_t \rho = -\delta\mathcal{F}/\delta\rho = 0$. Therefore, $\mathcal{F} = \frac{1}{2}|\nabla\rho|^2 + F(\rho)$ is conserved with respect to ρ. Since the boundary condition for ρ at the surface is $\partial_z \rho = 0$, and considering solutions with $\partial_x \rho = \partial_y \rho = 0$, then

$$\frac{1}{2}(\partial_z \rho)^2 + F(\rho) = F(\rho_0) \tag{15}$$

where ρ_0 is ρ at the surface.

The height of the granular layer h can be obtained by integration

$$\int_{-h}^{0} dz = \int_{1}^{\rho_0} \frac{d\rho}{\sqrt{2\left[F(\rho_0) - F(\rho)\right]}} \tag{16}$$

$$h = \int_{1}^{\rho_0} \frac{d\rho}{\sqrt{2\left[F(\rho_0) - F(\rho)\right]}} \tag{17}$$

The critical thickness h_s can be found by minimizing the integral over all values of ρ_0.

$$h_s(\delta) \equiv \min_{\rho_0} \int_{1}^{\rho_0} \frac{d\rho}{\sqrt{2\left[F(\rho_0) - F(\rho)\right]}} \tag{18}$$

In consequence, if $h > h_s(\delta)$, an stationary solution exists besides the static uniform solution.

3.2.2 Non-stationary case

For non-stationary solutions, small deviations from the uniform solid state $\rho = 1$ will be studied. For $\rho > 1$, the solid state is stable at small h, but it loses stability at a certain critical thickness h_c.

The most unstable modes of Eq. (13) satisfying the boundary conditions are in the form [6]

$$\rho = 1 - Ae^{\lambda t}\cos(\pi z/2h), \quad A \ll 1 \tag{19}$$

Replacing Eq. (19) into Eq. (13) gives

$$\lambda(h) = \delta - 1 - \pi^2/4h^2 \tag{20}$$

If $\lambda < 0$ the solid state is stable; if $\lambda > 0$, it is unstable. The critical stability occurs at $\lambda = 0$, which defines a critical height h_c

$$h_c(\delta) \equiv \frac{\pi}{2\sqrt{\delta - 1}} \tag{21}$$

The two critical heights $h_c(\delta)$ and $h_s(\delta)$ divide the $h - \delta$ parameter plane in three regions, Fig. 4. At $h < h_s(\delta)$, the trivial static equilibrium is the only stationary solution. At $h_s(\delta) < h < h_c(\delta)$, static equilibrium coexists with stationary flow. At $h > h_c(\delta)$, only granular flow exits. This picture agrees with experimental results.

4 Experiment

Avalanches on an inclined plane were studied experimentally by Daerr and Douady [9]. The experiment consists in a plane tilted at an angle φ to the horizon and covered by a flat layer of dry cohesionless glass beads (180 - 300 μm of diameter). The granular material is spread uniformly over the plane with an initial thickness h. The surface of the plane is made of velvet cloth, so the friction of the grains with it is much large than between themselves.

From dimensional analysis, the diffusion coefficient is $D = l^2/\tau$, where l is the characteristic length, which is of the order of the average grain diameter, and τ is the characteristic time, which is of the order of the collision time, $\tau \sim \sqrt{l/g}$. In order to compare theory with experiments, l is taken as the average grain diameter, $l = 240\ \mu m$, and $\tau = \sqrt{l/g}$.

The particles are initially at rest, and the experiment was performed by increasing the inclination angle φ slowly until an avalanche was triggered. Fig. (4) shows the theoretical predictions (lines) against the experimental results (symbols) for $h_c(\delta)$ and $h_s(\delta)$.

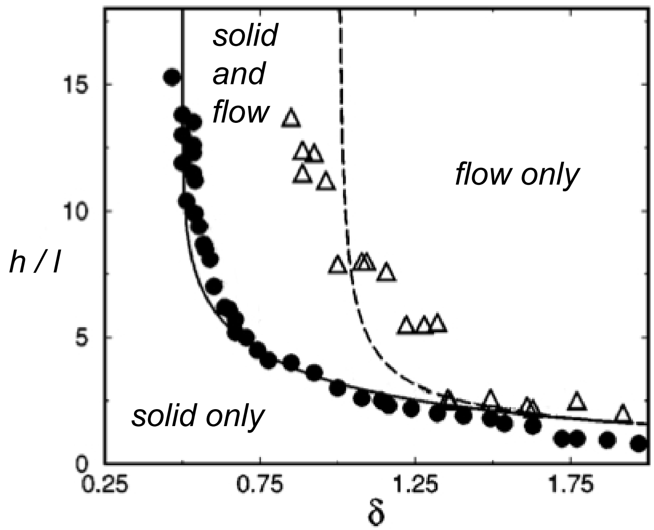

Figure 4: Dashed line corresponds to h_c, Eq. (21). Continuous line corresponds to h_s, Eq. (18). Symbols show experimental data [9]. The two asymptotic values $\delta = 0.5$ and $\delta = 1$ correspond to the experimental angles of $\varphi = 25°$ and $\varphi_1 = 32°$ respectively, from where the critical angle φ_0 was obtained.

5 Conclusion

A general continuum theory of dense granular flows was derived. In this model, an order parameter is introduced as a variable that controls the stability of the static and the fluidized phases. In particular, this theory is applied to the study of avalanches on an inclined plane configuration. This order-parameter model successfully explains many important phenomena observed experimentally [7]. For example, the existence of a bistable regime, where solid and fluid phases coexist.

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