Forest-Fires and Phase Transitions

Term Essay in Phase Transitions - PHYS 563

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Abstract

A large group of natural hazards (e.g. forest fires, earth quakes) are showing scaling law behavior even if they differ in size by magnitudes. As it turns out these class of systems can be described by Self-Organized-Criticality and it is indeed possible to relate them to the critical behavior of phase transitions. Using the example of forest-fires the question of how (well) the theoretical phase transition models fit the experimental results shall be studied.

Contents

1 Introduction

The Forest-Fire model belongs to the class of Self-Organized-Critical (SOC) systems, which are governed by a slow driving energy input and burst (avalanches) of dissipative outputs resulting often in a fractal structures. These systems were introduced by P. Bak et al. [2] in 1987 using the example of a sandpile model. These SOC models can be applied to many different fields, famous applications are for instance: earthquakes, solar flares, co-evolution, forest-fires and more. In addition they show scaling laws and are related to critical phenomena.
In this paper, starting from the context of self organized criticality the development of a forest-fire model will be explained, resulting in a prediction of key features and scaling behaviors for forest-fires. Although the obtained theoretical model was mostly studied using cellular automata simulations [1, 8, 4] it is possible to relate the theoretical results to experimental data using records forest-fires [7, 6, 3].

2 The Forest-Fire Model

In order to understand forest-fires, a theoretical model has to be established. The first model proposed by P. Bak et al. [1] will be described first. I will give a short overview about Self-Organized-Critical phenomena (SOC), since the described models are strongly connected to SOC. As it turns out however, the first proposed model obeys no criticality, which was shown by P. Grassberger et al. [5] and W.K. Mossner et al. [8] using cellular automate computer simulations. Adding an other degree of freedom however (a lightning probability) makes the system critical. This model will be described and scaling laws will be derived.
Let us start with Self-Organized-Critical systems.

2.1 Self-Organized-Criticality - SOC

In 1987 P. Bak et al. introduced the idea of Self-Organized-Criticality [2]. He stated that dynamical driven systems with many spatial degrees of freedom evolve to unstable states. These unstable states are dissipating their energy in bursts (avalanches). He also introduced a simple cellular automata model - the sandpile model - to understand the key features of SOC. In the sandpile model sand grains continuously are added to a grid. If the slope between two lattice sites exceed a certain threshold, the number at the larger site is reduced by four, while adding one grain of sand to each neighbor. He was able

to show that this model exhibits power law scaling in spatial and temporal distribution, e.g. the number of clusters $D(s)$ of size s scaled with some exponent τ: $D(s) \sim s^{-\tau}$. And for the distribution of lifetime he obtained $D(t) \sim t^{-\alpha}$.
In addition one can make an analogy to equilibrium critical phenomena [10]: Let us assume, that the current of incoming sand is the external magnetic field while the magnetization is given by the flow away from the a maximum. If the slope Θ is larger than a critical value Θ_C, the system is unstable and there will be a flow - hence a magnetization. This is the analogy to the ordered state. If $\Theta < \Theta_C$ no flow will be observed (disordered), unless an external field (stream of sand) is applied. This analogy helps to understand the system. It is however important to notice, that the physical principles are different from usual critical phenomena.
Summarizing his results, a system which exhibits SOC will be at a critical point over a large scale of external parameters. In addition it is important to notice that it is not necessary to tune a certain external parameter in order to obtain criticality, the system rather 'drives itself' towards the critical point - hence Self-Organized-Criticality.
Further explanations and scaling laws can be found [10, 2].

2.2 The P. Bak - Model

As mentioned before a SOC system consists of constant uniform energy input, the ability to 'store' this energy and dissipate it in bursts (avalanches) through spatial fractals. P. Bak et at. introduced a forest-fire model in order to demonstrate scaling and fractal energy dissipation [1]. The model consists of a cellular automata on a lattice with the following rules:

i Trees are randomly grown with a probability p at empty sites at each time step.

ii Trees, which are on fire will burn down at the next time step.

iii At the next time step, the fire will spread to all nearest neighbors.

In addition he introduced a correlation length $\xi(p)$ which basically states over which length scales, lattice site are correlated (we will later define it as the root of the mean squared distance from the center of mass of a forest cluster). The correlation length scales with an exponent ν as $\xi(p) \sim p^{-\nu}$. This relation can be understood, considering the following energy conservation argument [1]: The total number of grown trees are $L^d p$. Given the correlation length ξ, we know that the number of areas which are uncorrelated should be about $\frac{L^d}{\xi^d}$ (volume of system / correlated volume). Using the fractal dimension D, the number of trees burned in one domain(fractal) is ξ^D.

Since the number of grown trees should be equal to the number of burned trees we conclude:

$$L^d p = \frac{L^d}{\xi^d}\xi^D \Leftrightarrow \xi = p^{-\frac{1}{d-D}} \tag{1}$$

Hence $\nu = \frac{1}{d-D}$.
Given this scaling relation he expected a critical behavior for $p \to 0$ and argued that if the correlation length is larger than the system size, the fire will die out, while it will be sustained if the correlation length is smaller than the size of the system. This can be understood as follows: If the probability of growing a tree is small (large correlation length according to 1), we expect that the average time to grow a tree is larger than the average time to burn a tree. Clusters of trees will therefore be burned completely, hence the fire dies out. If on the other hand the probability of growing a tree is large, new trees can be grown at the boundary of tree clusters before the fire reaches these boundaries and the fire will be sustained.
The main results of this paper were, that this forest fire model can be regarded as a system at a critical point, which should yield scaling law behavior and that slow uniform energy input (grows of trees) result in fractal energy dissipation.

2.3 Why the Forest-Fire model by P. Bak et al. is not critical

As described in the previous section, P. Bak et al. suspected a critical behavior for $p \to 0$. Several computer simulations by P. Grassberger et al. [5] and W.K. Mossner et al. [8] however showed that the system is rather deterministic than critical in this limit. Their approach was as followed: First they observed that for large growth probabilities the system is dominated by many fire fronts, which building bubble like structures. These bubbles expand and merge with other bubbles to form nuclei for new bubbles. In addition they observed that this scenario was governed by a frequency proportional to $\frac{1}{p}$. Second, they found that if p is very small, the bubble structure vanished in favor of a regular spirals (see figure 1). Again the size and the distance between the spirals was of the order of $\frac{1}{p}$. This fact seemed to indicate a deterministic character of the system rather than a critical. In order to proof their claim, they looked at several properties of this system:

Spatial structure: The scaling of the number of fire sites with distance r from a fire site $D(r)$ could be shown to obey the scaling $D(r) \propto r^{D-1}$

proposed by P. Bak et al. However it appeared that P. Bak et al. studied accidentally an artificial 'horizontal' state, which was created by a poor choice of size and boundary conditions of the simulation and are presumably vanishing in the thermodynamic limit.

Temporal structure: In order to understand the temporal behavior of the system W.K. Mossner et. al. looked at the mean fire density as a function of time. They found that the mean fire density shows periodic behavior at a frequency f which gets more distinct for small p. This again is an indication of a deterministic system.
In addition they computed the temporal fire-fire-correlation function $G(\tau)$ and found that $G(\tau) \propto \cos(2\pi f \tau) e^{-\lambda\tau}$. Where $\lambda \propto p^{1.5}$ and $f \propto p^{0.9}$ in 2 dimensions. If one measures time in units of $1/f$ one obtains $G(\tau) \propto \cos(2\pi\bar{\tau}) e^{-p^{0.6}\bar{\tau}}$ with $\bar{\tau} = \tau f$. If $p \to 0$ the damping term will be approximately be one and one obtains an pure oscillation. This is another indication for a deterministic character of the system.

Scaling behavior: If the system is a critical point for $p \to 0$ it should be invariant under a transformation in length and time: a transformation in length and time yields the same system with a new growth probability p. This invariance should yield a scaling relation for the normalized probability $n(s)$ of finding a tree cluster of size s , $n(s) \propto s^{-b}$, where b is a transformation parameter (e.g. $x \to x' = bx$). Since the system is not exactly at $p = 0$ a crossover effect is expected in this scaling law if the cluster size exceeds a certain value s_{max}. An appropriate scaling function $\mathcal{C}$ can be introduced and the scaling relation becomes:

$$n(s) = \mathcal{C}\left(\frac{s}{s_{max}}\right) s^{-\tau} \tag{2}$$

First , they tried to collapse the data obtained by their simulations to verify the scaling relation. This was however not possible. In addition they tried to show that $s_{max} \propto p^{-\lambda}$ and that the radius of a forest fractal R scales as $R \propto s^{1/\mu}$. Both attempts failed. Furthermore they observed that $n(s)$ rather shows two distinct areas where the system behaves in a different manner.

Obtaining these results they concluded that the forest-fire model is not a self organized critical system. In addition they were able to come up with a deterministic model for the spiral state of the system, describing very well the observed behavior.
In the last sentence of this paper [8] they commented that adding a second parameter - a lightning probability f - to the system will make the system critical. This model will be described in the next section.

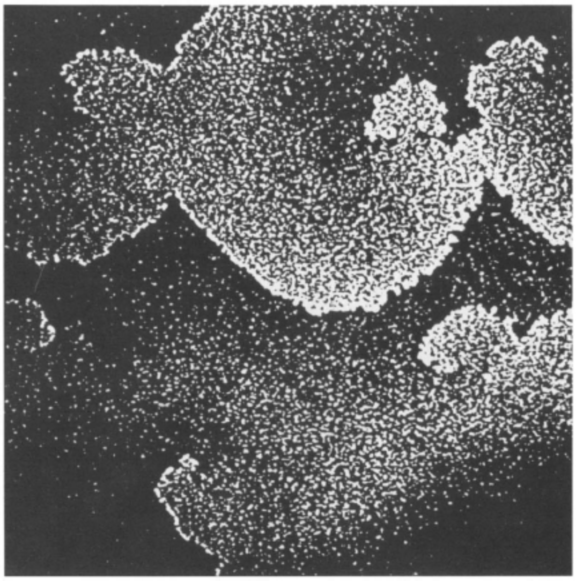

Figure 1: Fig. 1 from [8]: Steady state of the Bak-forest-fire model for $p = 0.005$ and $L = 1000$. The forest is black, white is empty. One can nicely see the spiral structure.

2.4 The Drossel-Schwabl-Model

In the previous section, we obtained that the forest-fire model proposed by P. Bak et al. is not critical, because the system behaves deterministic. We would have expected to observe all sizes of fires up to the correlation length if the system would have been critical. Instead it was found that there is a typical length $(1/p)$, which determines the size of fires. Let us try to understand why this is so [4]: First, consider a small cluster of trees which is not burning. This cluster will grow until it merges with a cluster which is on fire. Hence there is a threshold (given by a specific length) for a cluster to actually catch fire. To determine this length scale, one has to remember that the fire burns constantly in the steady state. This means that only these tree clusters will burn constantly which are growing fast enough. And since the growth of a cluster is proportional to its circumference, only cluster larger than a certain size (actually with a diameter of about $1/p$) will burn constantly. This argument shows that there are only fires of a certain size. To obtain criticality we would want to have fires of all sizes. This is the reason why B. Drossel et al. introduced a lightning probability f to the system, which will ignite smaller forest clusters [4]. The forest-fire model therefore consist of the rules mentioned above and the additional rule:

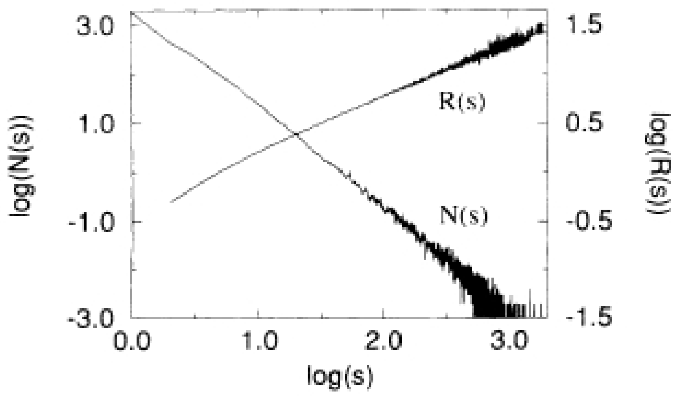

Figure 2: FIG. 2 from [4]: Average number of tree clusters and average cluster radius as function of cluster size in $d = 2$ and $f/p = 1/70$. The simulations yield $tau = 2$ and $\mu = 2$

iv A tree which is not burning (and has no burning neighbors), will catch fire with a probability f.

First let us try to understand the behavior of this system by assuming that a forest cluster which is on fire will burn down immediately. In this case we can understand that the system is fully described by the fraction f/p: If the lightning probability is increased by a factor, more trees will burn on average. To obtain a system with the same dynamics the growth probability has to be increased by the same factor. Hence, there should be only one important parameter f/p. Now, let $\bar{\rho}$ denote the average tree density and $\bar{s}$ denote the average number of trees destroyed be a lightning stroke, i.e. the average size of a tree cluster. $\bar{s}$ can be expressed as:

$$\begin{aligned} \bar{s} &= \frac{\text{average number of trees}}{\text{average number of lightning strokes}} = \frac{\bar{\rho}_{empty}\, L^d\, p}{\bar{\rho}\, L^d\, f} \\ &= \left(\frac{f}{p}\right)^{-1} \frac{(1-\bar{\rho})}{\bar{\rho}} \end{aligned} \tag{3}$$

This relation represents a power law for $\bar{s} \propto (f/p)^{-1}$ in the limit $(f/p) \to 0$ as long as $\bar{\rho} < 1$ in this limit.

As we have seen, we expect certain scaling behavior. Can we use this result for the original forest fire model? The answer is yes, if the time which it takes to burn a cluster of size s, $T(s)$ is much smaller than the time to add (grow) new trees to the cluster, which is proportional to $1/p$. Hence as

long as $T(\bar{s}) << \frac{1}{p}$ our results should still be valid and critical behavior is expected.
Let us now get a more precise by defining $R(s)$ as the square root of the mean quadratic deviation from the center of mass of a cluster of size s. Since trees in one cluster are correlated, it makes sense to define the correlation length as $\xi = R(\bar{s})$. It follows that $\xi \propto (f/p)^{-\nu}$. In addition B. Drossel et al. are defining the number of tree clusters of size s between $[s, s+ds]$ as $N(s)ds$.
Under a length rescaling $x \to x/b$ the functional form of $N(s)$ and $R(s)$ should not change, i.e. is invariant - this means a 'coarse grained' system looks essentially the same and one can follow the following scaling laws:

$$N(s) \propto s^{-\tau} \times \begin{cases} \mathcal{C}\left(\frac{s}{s_{max}}\right) & \tau > 2 \\ \mathcal{C}\left(\frac{s}{s_{max}}\right) \log^{-1}(s_{max}) & \tau = 2 \end{cases} \tag{4}$$

$$R(s) \propto s^{1/\mu}\, \tilde{\mathcal{C}}\left(\frac{s}{s_{max}}\right) \tag{5}$$

where $s_{max} \propto (f/p)^{-\lambda}$ is the maximum size of a cluster.
$\mathcal{C}(x)$ and $\tilde{\mathcal{C}}(x)$ are scaling functions with:

$$\mathcal{C}(x) = \begin{cases} 1 & x << 1 \\ 0 & x >> 1 \end{cases} \tag{6}$$

similar for $\tilde{\mathcal{C}}(x)$.
Using equation 1, 5 and the definition of the correlation length one can deduce:

$$\lambda = \nu\mu \qquad \text{and} \qquad d = \mu(\tau - 1) \tag{7}$$

For instance: The volume of the system L^d is :

$$\begin{aligned} L^d &\sim \int ds\; R(s)^d\; N(s) \sim \int ds\; s^{d/\mu}\; s^{-\tau}\; \mathcal{C}(s/s_{max})\; \tilde{\mathcal{C}}(s/s_{max}) \\ &= s_{max}^{1-\tau+d/\mu} \int dx\; x^{d/\mu-\tau}\; \mathcal{C}(x)\; \tilde{\mathcal{C}}(x) \end{aligned} \tag{8}$$

The integral is just a constant and since $s_{max} \to \infty$ for $(f/p) \to 0$ one has to assume $1 - \tau + d/\mu = 0$.
Using the relations above and the fact that:

$$\bar{s} = \frac{\int ds\; s^2 N(s)}{\int ds\; sN(s)} \tag{9}$$

one can derive all exponents in the limit $T(\bar{s}) << \frac{1}{p}$ and obtains:

$$\lambda = 1 \qquad \tau = 2 \qquad \mu = d \qquad \nu = 1/d \tag{10}$$

(for detailed calculations see [4]).
Now, can one actually observe these scaling relations? Many computer simulations were performed and the exponents and scaling relations could indeed be verified (see figure 2).

3 Experimental Results

We now have a good understanding about the theory and models of forest-fires. However it is reasonable to ask whether the found properties of the theoretical model apply to real forest-fires. First, one has to ask what quantities can be measured in real forest-fires? The most obvious quantities are the size s of a forest-fire and its relative occurrence. These two quantities are related trough equation 4.
This relation was studied amongst others by B. Malamud et al. [7], by C. Ricotta et al. [9] and by A. Corral et al. [3].
C. Ricoota et al. used a wildfire catalog consisting of 9164 forest fires form 1986 to 1993 in north Italy to verify the relation between occurrence and size of fires. Their results are shown in figure 3. They were able to fit a scaling law over a many orders of magnitude and obtained a critical exponent $\tau \approx 1.446/2$.
However their scaling behavior differs for small and large fire sizes: They argue that is due to the non-homogeneous landscape and the human intervention, which reduces the occurrence of large fires. In addition they argue that many small fires might not have been recorded. They conclude that the found scaling behavior is valid over large ranges and note that the ignition mechanism of forest fires are more complicated, since 90% of these fires were caused by humans.

Similar analysis was done by B. Malamud et al. in 1998. They looked at several datasets, acquired in landscapes with different structure and climates. Their results are shown in figure 4. They obtained very good scaling behavior and exponents in the order of $1.31 - 1.49$. In addition they followed that given enough data about small and medium fires it is possible to make statements about the average size of fires up to 50 years. Furthermore they did computer simulation and observed a finite size effect for small values of f/p: The number of large fires increases, since the forest clusters spread nearly the whole lattice and large clusters are dominant. This effect could

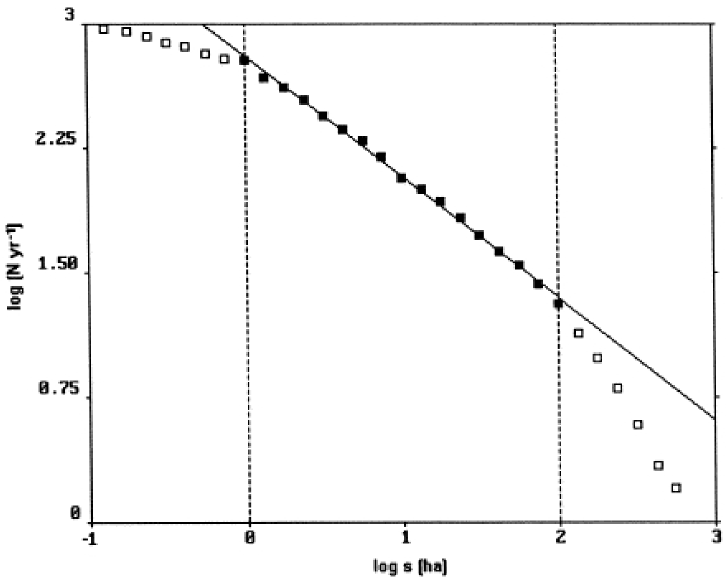

Figure 3: Fig. 1 from [9]: Number of wildfires in north Italy from 1986-1993 from [9]. The regression yields a fractal dimension D=1.446.

actually be mapped to a real phenomena - the 'Yellowstone effect'. Since the Yellowstone National Park had a strictly policy of suppressing fires (small lightning probability f) until 1972, a large number of dead trees accumulated which resulted in huge fires in 1988.

Lastly, I will discuss experimental results for the occurrence of forest-fires in time. This was extensively studied by A. Corral et al. [3]. They studied the spatial occurrence as well as the temporal occurrence at forest fires recored in Italy and obtained similar results found by C. Ricotta et al. for the spatial case. However the temporal case is interesting since data collapse is observed nicely. First they defined a waiting time probability density between to fires $D(\tau, s_c)$, where τ is the time and s_c a threshold (i.e. only fires larger than this threshold are considered). The results are shown in figure 5 (a) As it turned out the slope of the waiting time probability density depends upon the threshold. However rescaling by the mean rate of fire occurrence $R(s_c)$ yield:

$$D(\tau, s_c) = R(s_c)\ F(R(s_c)\tau) \tag{11}$$

where $F(x)$ denotes a new scaling function. Plotting D/R versus $R\tau$ collapsed the data (see figure 5 (b)). This collapse means that the waiting time has the same shape but on different scales. But why do we observe scale invariance in time? The answer is as follows: Define the number of fires larger

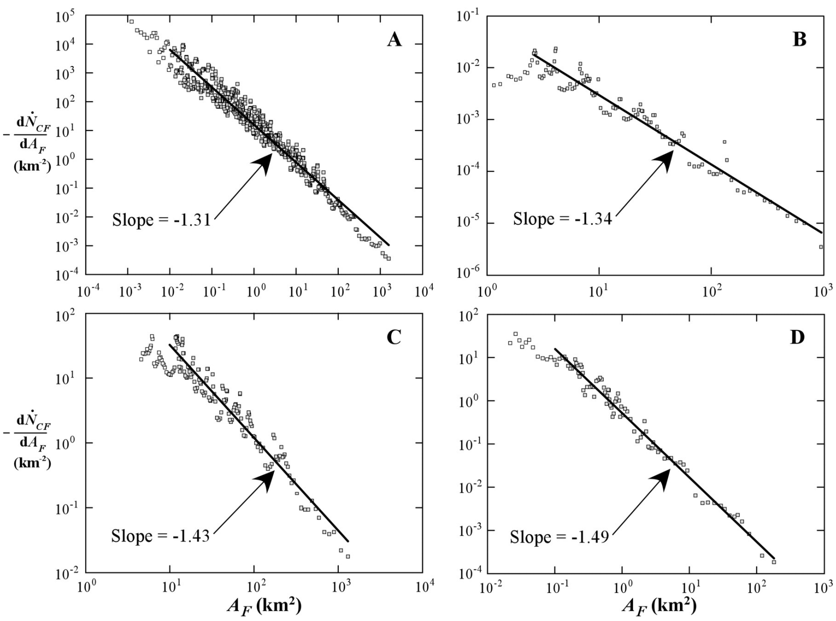

Figure 4: Fig.2 from [7]: Noncumulative frequency-area distributions for forest-fires for different regions A,B,C,D. See [7] for further information.

than s_c per unit time - the instantaneous rate as $r(t, s_c)$. The waiting time density reads than as:

$$D(\tau, s_c) = \frac{1}{R(s_c)} \int dr\ D(\tau, s_c|r)\ P(r) \tag{12}$$

where $D(\tau, s_c|r)$ is the conditional probability density of having a waiting time at time t given a rate r. The probability $P(r)$ of having a rate r can be expressed as $P(r) = r\ \rho(r, s_c)$ where ρ is the rate density. Since the probability distribution of fire sizes $D(s)$ is stationary (does not change with time) a transformation of the rate $r(t, s_c) = P(s > s_c|s > s_0)\ r(t, s_0) = p\ r(t, s_0)$ will yield a transformation of the density as $\rho(r, s_c) = \frac{\rho(r/p, s_c)}{p}$. Hence the density ρ obeys a scaling law $\rho(r, s_c) = f(r/p)/p$ with some scaling function $f(x)$. Equation 12 reads as:

$$\begin{aligned} D(\tau, s_c) &= \frac{1}{R(s_c)} \int dr\ r^2\ g(r\tau)\ f(r/p)\ p^{-1} \\ &= \frac{p^2}{R(s_c)} F(p\tau) = R(s_c) \tilde{F}(p\tau) \end{aligned} \tag{13}$$

with some scaling functions $F(x)$ and $\tilde{F}(x)$. In the first line it was assumed that $D(\tau, s_c|r) = r\ f(r\tau)$ obeys a scaling relation if the fire-size distribution

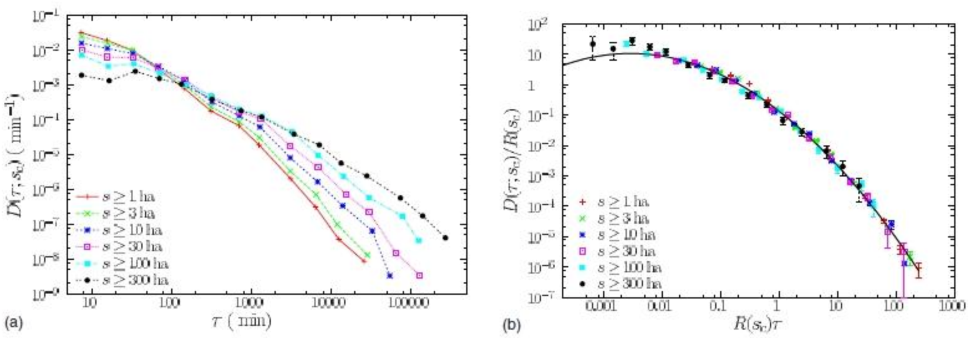

Figure 5: FIG. 3 from [3] (a) Probability density distribution for different cutoff sizes s_c. (b) Data collapse.

is stationary. And $R(s_c) = pR(s_0)$
Summarizing a scaling for the instantaneous waiting time yield to non-instantaneous waiting times, if the size distribution is stationary.

4 Conclusion

As a conclusion, we have seen that a well studied and understood theoretical model in the context of self-organized-criticality exists. SOC systems are usually characterized by a uniform energy input and dissipative avalanche (fractals) and can be described using scaling theory. The theoretical model can make statements about scaling behavior which can be observed in nature. Not only for the reason of modeling real forest-fires, but also for the purpose of understanding self-organized-criticality and its relation to basic concept of nature, the forest-fire model is an interesting system to study.

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