# Riemannian geometry of critical phenomena

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## ABSTRACT

In this review, it is discussed how by incorporating the theory of fluctations into the basic axioms of thermodynamics, thermodynamic systems can be mapped into appropriate Riemannian geometries. For systems such as the ideal gas, ferromagnetic one-d Ising model and the Van der Waals Gas, it is found that the curvature of this manifold is related to the correlation volume of the system.

The correlation volume is conventionally calculated from a statistical "microscopic" description of systems. For other systems such as the antiferromagnetic model and Kerr Newmann black holes the correlation volume is given a new interpretation. In the case of Kerr-Newmann black holes it is interpreted as the average number of correlated planck areas at the hole's surface.This idea may be formally extended to understand phase transitions in black holes thermodynamics where there is not a completely developed microscopic theory "quantum gravity".

# 1 INTRODUCTION

Black holes are objects of interest both theoretically and experimentally. They represent an extreme system where the gravitational force reigns supreme. Black holes are a prediction of general relativity and it is widely believed that there is one at the center of every galaxy. On the astronomical scale, black holes are believed to play important roles in the large scale structure of space and time, playing a major role in galaxy formation. Theoretically black holes are of huge interest because close to their centres, the curvature of space time and energy density of matter are infinite according to Einstein's equations of classical gravity. It is expected that the laws of quantum gravity should hold under this conditions, therefore an understanding of black hole interactions is crucial to a unified view of the two pillars of modern phyics (quantum physics and GR). In this respect, there is still quite some work to be done both theoretically and experimentally. Also, in the modern context of ADS/CFT black hole models are crucial to the understanding of the holographic principle.

The goal of this work is to present an approach that has been applied and tested on better understood systems like the ideal gas and the 1-d Ising Model. This approach is based on Riemannian Geometry on the phase space of thermodynamic variables. The original approach due to George Ruppenier[1] is based on a metric constructed from the second derivative matrix of the entropy. Up to second order in fluctuations of thermodynamic variables from equilibrium, the Ruppeiner metric physically measures the likelihood of fluctuations from equilibrium. Based on calculations done on the classical ideal gas model, an equation similar to that of general relativity is postulated to relate the curvature of the thermodynamic geometry to the interaction strength of the physical system. For the 1-d antiferromagnetic Ising model, Van der Waals gas, this postulate further suggests that the interaction strength is proportional to the correlation volume of the system which is conventionally calculated in statistical mechanical treatments as giving the range of the correlation function $G(r)$. For systems such as the 1-d ferromagnetic model a new but useful interpretation of correlation length has to be adopted . This new intepretation is extended to the Ruppiener geometry for kerr-Newmann black holes.

Other approaches to thermodynamic geometries are also possible. In particular by appealing to invariance of thermodynamics under a change of thermodynamic potentials, a Legendre invariant geometry for the study of black holes physics reveals the points at which the heat capacity at constant charge and angular momentum diverges. According to a classiification of black hole phase transitions due to Davies [7], these are also the phase transition points of black holes. This classification is not generally agreed upon and other classifications based on for instance a change in topology have been devised. These are not discussed in here, rather attention is devoted to the formalism of the geometric formulation of thermodynamics. The results obtained from an application to black hole thermodynamics and criticality are also summarized and discussed.

# 2 GEOMETRIC FORMULATION OF FLUCTUATION THEORY

The goal of a geometric formulation of thermodynamics is to provide a representation of systems which is intrinsic and does not depend on any particular choice of variables. In the case of fluctuation theory, the main foundation of a geometric formulation are that the equilibrium states of a thermodynamic system can be represented by points on an n-manifold. Where n is the number of independent physical quantities required to fully specify the state of the system, for axample n = 2 for an ideal gas. This manifold is assumed differentiable except at phase transitions and critical points [1].

The second foundation builds on an understanding that the equilibrium value of thermodynamic quantities fluctuates about a mean value a result which follows from statistical mechanics [9]. According to this theory given an open system $A_v$ with fixed volume V immersed in a large energy reservoir, the values of thermodynamic quantities x of $A_v$ will fluctuate around a most probable value y with a probabilty distribution $W(x,y)dy = Ce^{S(x,y)/K_B}$, where S(x,y) is the total entropy, $K_B$ is Boltzmann's constant and B is a normalization constant. This is really just a consequence of the way entropy is usually defined. As shown in [1], for a pure fluid the W(x,y) can be expanded around its maximum x up to second order to give a second moment:

$$W(x,y)dxdy = 2\pi \exp(-\frac{1}{2}g_{ij}\Delta y_i \Delta y_j)\sqrt{g}(x)dxdy \tag{1}$$

Where $g_{ij}(x) = -\frac{\partial^2 S}{\partial y_i \partial y_j}$. Since the entropy is maximixed at $x$, $g_{ij}(x)$ is positive definite and is thus a natural candidate for a metric on the 2-manifold of a fluid system with line element:

$$ds^2 = g_{ij}\Delta y_i \Delta y_j \tag{2}$$

The physical interpretation of this line element is that likely configurations are closer to the mean configuration while unlikely ones are further apart. It is well known in Riemannian and Pseudo-Riemmanian geometry that a metric tensor leads to the definition of invariants related to the curvature of a manifold. This fact plays a very important role in general relativity [5]. The curvature is an intrinsic property of the manifold and it is natural to enquire what the curvature of the fluctuating metric would mean for thermodynamic systems. A guide to a general answer to this question is given in [1] where the metric in temperature, density $(T,\rho)$ coordinates for a classical ideal gas given by:

$$ds^2 = \frac{C_v}{k_B T^2}dT^2 + \frac{V}{k_B T \rho^2 K_T}d\rho^2 \tag{3}$$

Where $C_v$ and $K_T$ are the heat capacity at constant volume and isothermal compressibility respectively. The equation of state is $P = \rho k_B T$ and $C_v = Nf(T)$. By making a change of coordinates $y_1 = (2V\rho)^{\frac{1}{2}}(cos\frac{\tau}{2} + sin\frac{\tau}{2})$ and $y_2 = (2V\rho)^{\frac{1}{2}}(cos\frac{\tau}{2} - sin\frac{\tau}{2})$ where $\tau = \int_C^T (\frac{f(T)}{k_B T^2})^{\frac{1}{2}}\, dT$ and C is an arbitrary constant positive temperature, one gets that the line element for an ideal gas reduces to $ds^2 = dy_1^2 + dy_2^2$ which is just the line element of 2-d euclidean space with zero curvature.

The main feature of an ideal gas is the absence of interactions therefore the result that the ideal gas thermodynamic geometry has zero curvature suggests an association of curvature with interaction strength. In [1] a relation is postulated as given by:

$$VK(x) = qI(x) \tag{4}$$

This is the fundamental equation of this geometry, and $q$ is a constant that can only be determined from experiment.

$$K(x) = \frac{-1}{2g^{\frac{1}{2}}}\left[\frac{\partial}{\partial T}\left(g^{\frac{-1}{2}}\frac{\partial g_{\rho\rho}}{\partial T}\right) + \frac{\partial}{\partial \rho}\left(g^{\frac{-1}{2}}\frac{\partial g_{TT}}{\partial \rho}\right)\right] \tag{5}$$

is the Gaussian curvature , $g_{TT} = \frac{C_v}{K_B T^2}$, $g_{\rho\rho} = \frac{V}{K_B T \rho^2 K_T}$, $g = g_{TT} g_{\rho\rho}$ and $I(x)$ is an object that characterizes interaction strength. Using the scaling relationships for a fluid along a path of critical density $\rho_c$, $\frac{C_v}{V} = At^{-\alpha}$ and $K_T = Bt^{-\gamma}$, where A and B are constants and t is the reduced temperature one obtains on using the experimentally calculated $\alpha = 0.1$ and $\gamma = 1.19$ [1]

$$\frac{-V}{2g^{\frac{1}{2}}}\frac{\partial}{\partial T}\left(g^{\frac{-1}{2}}\frac{\partial g_{\rho\rho}}{\partial T}\right) = 0.21\frac{k_B}{A}t^{\alpha-2} \tag{6}$$

A more difficult calculation[1], yields a term that is negligible and from equation (4), it can be concluded that $I(\rho_c, t) = 0.21\frac{k_b}{qA}t^{\alpha-2}$. Therefore I(x) has the same dimensions and critical exponent as $\xi^3$ where $\xi$ is the correlation length. This suggests an identification of I(x) with $\xi^d$ for any physical system. We note that this is merely an hypothesis whose validity can only be tested with experimentally determined values of correlation volumes for well known systems. In the next section a summary of this hypothesis as applied to the 1-d ising model is given.

# 3 RESULTS FOR 1-D ISING MODEL

By the same reasoning applied above, the metric for the 1-d ising model is given by $ds^2 = \frac{1}{T}\frac{\partial S}{\partial T}dt^2 + \frac{1}{T}\frac{\partial H}{\partial M}dM^2$, where $H$ is the external field and $M$ is the magnetization. The Gaussian curvature $K(x)$ is given an expression analogous to (5). To test the hypothesis that $K(x)$ obtained from geometry is directly related to the correlation volume, a numerical calculation on the partition function for the 1-d ising hamiltonian has been done [10]. For the ferromagnetic ising model with coupling constant $J > 0$ it is found that the correlation length $\xi_G$ obtained from geometry is in excellent agreement with the known values of $\xi$ which gives the range of spin-spin correlations $G(r)$ never deviating by more than one lattice constant [10]. When $J = 0$, the manifold of thermodynamic states is effectively one dimensional which implies a curvature of zero, agreeing with the expectation that curvature is proportional to interaction strength. For the antiferromagnetic case with $J < 0$, the results are not quite accurate in comparison with the usual definition of the correlation length. Nevertheless in [10] it is is argued that $\xi_G$ can be interpreted physically as given the average length due to interactions of clusters of alligned spins.

For systems such as the Van der Waals gas and ideal paramagnet there is a similar agreement between the curvature and the correlation length. The thermodynamic curvature has been extensively discussed in the literature, see [11] for a review.

Taking this formulation as containing some useful physics, this geometric model of analyzing the thermodynamics of systems can lead to insight about phase transitions is black holes where very little is understand about the microscopic processes that govern the interactions in black holes. In the next sextion, a quick review of black hole thermodynamics is given.

# 4 REVIEW OF BLACK HOLE THERMODYNAMICS

According to Einstein's theory of General relativity the gravitational field manifests itself as curvature of space and time due to sources of energy and momentum. In [5], it is demonstrated that an analysis of Einstein field equations combined with the thermodynamics of stars reveal that the final state of collapse of a star about eight time more massive than our sun results in objects known as black holes. When a spherically symmetric collapsing star of mass M shrinks beyond a radius of $\frac{2GM}{c^2}$ the gravitational field becomes too strong that even light cannot escape this radius, and a black hole is inevitably formed. This is known as a Schwarschild black hole and is uniquely specified by its mass. In this case the radius $\frac{2GM}{c^2}$ is known as the event horizon. In the more general case after the collapse of a star state of the black hole formed is uniquely determined by its mass $M$, angular momentum $L$ and charge $Q$, that this is the case is demonstrated in [5]. This is known as a Kerr-Newmann black hole. Another black hole of interest is the BTZ black hole which is the black hole solution to Einsteins equations in 2+1-dimensions with negative cosmological constant. However this is also specified by its mass, charge and angular momentum. This state of affairs i.e characterization by three parameters$(M, L, Q)$ leads to a thermodynamic representation of black holes.

Since the final state of a black hole is uniquely determined by three parameters regardless of the structure of the star that resulted in the hole, a given configuration of (M,L, Q) can be a result of several microstates [7]. This idea leads to the notion of black hole entropy. The details are ommitted here, but calculations from pure classical considerations give the entropy of a black hole as infinite a nonsensical result that was rescued by Steven Hawkings [2]. A true understanding of Hawking's result involves the complicated language of quantum field theory in curved space-time, but the resulting entropy is given in natural units by

$$S = \frac{A}{4} = \pi(2M^2 - Q^2 + 2\sqrt{M^4 - M^2Q^2 - L^2}) \tag{7}$$

where A is the surface area of the event horizon. The second law of black hole thermodynamics is that in any process involving black holes the total surface area can never decrease [7]. A major issue in black hole thermodynamics is that of the definition of temperature, with details ommited here, the temperature of a black hole is defined to be $T = \frac{\kappa}{2\pi}$, where $\kappa = 8\pi\frac{\partial M}{\partial A}$ is known as the surface gravity. This definition is essentially a zeroth law, and from equation [7] and the definitions given, the first law takes the form.

$$dM = TdS + \Omega dL + \Phi dQ \tag{8}$$

Where

$$T = \frac{\partial M}{\partial S}, \Omega = \frac{\partial M}{\partial L}, \Phi = \frac{\partial M}{\partial Q} \tag{9}$$

are the temperature, angular velocity and electric potential respectively.

It must be emphasized that despite the fact that these thermodynamic notions are well defined, the internal structure of black holes is still poorly understood. In particular it is not still quite clear what is precisely meant by a phase transition in a black hole. Of course a full understanding can only be achieved by a well developed theory of quantum gravity. Nevertheless, a lot of work has been done in this regard. From equations (9,10), the heat capacity of a black hole at constant charge and angular momentum, i.e a measure of the energy transfered in a process between two equilibrium states that only changes the mass is given by:

$$C_{L,Q} = T\left(\frac{\partial S}{\partial T}\right)_{L,Q} = \frac{8MS^3T}{L^2 + \frac{Q^2}{4} - 8T^2S^3} \tag{10}$$

Davies [7] based his classification of phase transitions in black holes on this heat capacity. This is not generally agreed upon and this issue re-surfaces in the geometric formulation as well.

# 5 FLUCTUATION GEOMETRY FOR BLACK HOLES

As reviewed in the previous section a Kerr-Newman black hole is characterized by its mass M, angular momentum L and charge Q. Also from the formula for the entropy given in (7), a fluctuation geometry based on the entropy similar to that for the ideal gas for black hole thermodynamics can be easily obtained. Before this is explored, a subtlety involved in the formalism must be adressed. We note that other thermodynamic geometries can be obtained based on the hessian matrix of other thermodynamic potentials. A common one is Weinhold geometry based on the internal energy [8]. It is a well known fact from thermodynamics that the physical description of systems does not depend on the thermodynamic potential being used. Since different potentials are related by a legendre transformation, the authors of [7] argued that any thermodynamic metric that represents an intrinsic description must be legendre invariant.

It turns out that the Ruppiener geometry is not Legendre invariant and this has led to a formulation of a legendre invariat geometry for black holes. Here both results from the Ruppiener geometry and Legendre invariant geometry are summarized.

In [8], after some differential geometric and physical considerations. It is obtained that the legendre invariant metric on the phase space of thermodynamic varaibles for black holes is given by:

$$ds^2 = (MS_M + QS_Q + LS_L)(S_{MM}dM^2 - S_{QQ}dQ^2 - S_{LL}dL^2 - 2S_{QL}dQdL) \tag{11}$$

Where subscripts denote partial differentiation.

Using the entropy formula (7), and introducing coordinates $(x^1, x^2, x^3) = (M, J, Q)$, the line element for the ruppiener geometry based on the hessian of the entropy is given by:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{12}$$

where $g_{\mu\nu} = -\frac{8\pi}{L_p^2}\frac{\partial^2 S}{\partial x^\mu \partial x^\nu}$ and $L_P$ is the planck length.

Next it is explored how this different formulations may yield insight into critical phenomena in black holes.

# 6 CRITICAL PHENOMENA IN BLACK HOLES

## 6.1 Legendre invariant geometry

### 6.1.1 Reissner-Nordstorm black hole

This is a black hole with L = 0, i.e no angular momentum. It is spherically symmetric with two horizons at $r_\pm = M \pm \sqrt{M^2 - Q^2}$ From (7), the entropy is $S = \pi(M + \sqrt{M^2 - Q^2})^2$, the heat capacity (10) in terms of $r_\pm$ is $C_Q = -\frac{2\pi^2 r_+^2 (r_+ - r_-)}{r_+ - 3r_-}$ The scalar curvature in this case for the metric (11) is given by:

$$R = \frac{(r_+^2 - 3r_- r_+ + 6r_-^2)(r_+ - r_-)^2}{\pi^2 r_+^3 (r_+^2 + 3r_-^2)^2 (r_+ - 3r_-)^2} \tag{13}$$

In the extremal limit which is a zero temperature black hole, the scalar curvature is zero. From the the expression (13), its is deduced that the scalar curvature diverges at $r_+ = 3r_-$ which is precisely where the heat capacity diverges signalling a second-order phase transition.

### 6.1.2 Kerr black hole

This is a neutral black hole corresponding to Q = 0. It represents a stationary, axially symmetric, rotating black hole with two horizons located at: $r_\pm = M \pm \sqrt{M^2 - L^2/M^2}$, the entropy (7) is $S = 2\pi(M^2 + \sqrt{M^4 - L^2})$, the heat capacity (10) in terms of $r_\pm$ is given by $C_L = \frac{2\pi^2 r_+ (r_+ + r_-)^2 (r_+ - r_-)}{r_+^2 - 6r_+ r_- - 3r_-^2}$. The scalar curvature obtained from the legendre invariant metric is given by:

$$R = \frac{(3r_+^3 + 3r_+^2 r_- + 17 r_+ r_-^2 + 9r_-^3)(r_+ - r_-)^3}{2\pi^2 r_+^2 (r_+ + r_-)^4 (r_+^2 - 6r_+ r_- - 3r_-^2)^2} \tag{14}$$

In this case the scalar curvature is found to diverge when $r_+^2 - 6r_+ r_- 3r_-^2 = 0$ which is exactly where the heat capacity diverges signalling a second-order phase transition.

### 6.1.3 General Kerr-Newman black hole

This represents the most general rotating and charged black hole with horizon at $r_\pm = M \pm \sqrt{M^2 - \frac{L^2}{M^2} - Q^2}$ . The scalar curvature obtained in this case is $R \propto \frac{1}{D}$ where

$$D \propto A(M, Q, L)[2M^6 - 3M^4 Q^2 - 6M^2 L^2 + Q^2 L^2 + 2(M^4 - M^2 Q^2 - L^2)^{\frac{3}{2}}]^2 \tag{15}$$

and $A(M,Q,L)$ is always positive when $M^4 \geq M^2Q^2 + J^2$ as required by the cosmic censorship hypothesis (which is a limit proposed by Roger Penrose to avoid the presence of naked singularities in our universe [7]). The term in the squared brackets is the denominator of the heat capacity, therefore one concludes that the heat capacity diverges when the curvature scalar diverges, also signalling a second order phase transition.

## 6.2 II. Ruppiener geometry

As explained above, concerns have been posed about the legendre invariance of the Ruppiener metric. Nevertheless as discussed in the first few sections there are hints that there is some physics to be learnt from the Ruppiener metric. It is explained in [4] how a different interpretation of the curvature in terms of the number of correlated Planck lengths at the surface of the black hole is possible. Since the issue of phase transitions in black holes is not really well understood, this contrasting point of view is summarized here. In terms of the entropy the first law (8) implies $\frac{1}{T} = (\frac{\partial S}{\partial M})_{L,Q}, -\frac{\Omega}{T} = (\frac{\partial S}{\partial L})_{M,Q}, -\frac{\Phi}{T} = (\frac{\partial S}{\partial L})_{L,M}$. Defining $(\alpha, \beta) = (\frac{L^2}{M^4}, \frac{Q^2}{M^2})$ and $(K,P) = (\sqrt{1-\alpha-\beta}, \sqrt{1+\alpha})$. The temperature is easily shown to be $T = \frac{4K}{(K^2+2K+P^2)M}$ and the cosmic censorship hypothesis imply $\alpha + \beta < 1$. Also for later use, we define:

$A = -2K^3 - 3K^2 - 2P^2K + 2K - 3P^2 + 4$ and $B = K^3 + P^2K - K + 1$

Since the Ruppiener geometry is based on fluctuations away from equilibrum, several cases as analyzed in [4] are now summarized.

### 6.2.1 M, Land Q fluctuating

When M, L and Q, fluctuates as demonstrated in [7], the black hole is unstable and it is not appropriate to use a fluctuation theory based on second order fluctuating moments here.

### 6.2.2 M fixed, L, Q Fluctuating

In this case the metric corresponds to a 2-d geometry and the curvature is given in terms of the variables defined above by:

$$R = \frac{K^5 + P^2K^3 - 2K^3 - 2K^2 + 3P^2K + 2}{4\pi K B^2} \times \left(\frac{M_p}{M}\right)^2 \tag{16}$$

where $M_p$ is the planck mass. It turns out that these curvature never diverges in the physical regime allowed by cosmic censorship[4].

### 6.2.3 L fixed, M and Q fluctuating

The metric for this case is also a 2-d geometry with scalar curvature:

$$R = \frac{1}{2\pi K A^2} f(K,P) \times \left(\frac{M_p}{M}\right)^2 \tag{17}$$

Where $f(K,P)$ is a polynomial function in its arguments and hence has no irregular behaviour. The curvature diverges only in the extremal limit $K = 0$.

### 6.2.4 Q fixed, M and J fluctuating

The scalar curvature in this case is: $R = \frac{1}{2\pi K A^2} g(K,P) \times \left(\frac{M_p}{M}\right)^2$ Where $f(K,P)$ is a polynomial function in its arguments and hence has no irregular behaviour. The curvature diverges only in the extremal limit $K = 0$ and also when $A = 0$. In the latter case the heat capacity $C_{\Omega,Q}$ diverges as $A^{-2}$.

# 7 DISCUSSION AND CONCLSUIONS

As described in this review, there is evidence that by incorporating fluctuations into thermodynamics, a fluctuation geometry which can lead to insight into critical phenomena in black holes can be developed. The curvature scalar associated with a geometry based on the second derivative of the Hessian matrix of the entropy found for systems such as the ferromagnetic 1-d Ising model and the classical ideal gas is found to be proportional to the correlation volume. The correlation length is in accord with that usually defined as the typical length scale of spatial correlations $G(r)$ derived from statistical mechanical models. In the case of the antiferromegnetic 1-d ising model, this is not quite the case and a new interpretation of the correlation length as the average length of correlated spins has to be adopted. This is particulalry relevant in the application to Kerr-Newmann black holes where the correlation area obtained for the Ruppiener Geometry is interpreted as the average number of correlated planck areas at the surface of a black hole. This interpretation is of course speculative and can only be confirmed by a microscopic theory/experimental data of black holes.

The physical interpretation of the Ruppiener metric is that it measures the likelihood of fluctuations from equilibrium to second order. It is not quite clear how higher order flcutuations could be incorporated into the theory. Also there seems to be no real physical motivation for this formulation of thermodynamics, only that it seems to predict a way of understanding correlation lenghts and predicting criticality. In particular its is not well understood how this method is related to the more standard renormalization group technique for studying critical behaviour.

By invoking legendre invariance i.e the well known fact that thermodynamics does not depend on the choice of potential chosen to study a system, legendre invariant geometries similar to the Ruppiener geometries can be developed. For Kerr-Newmann black holes, the scalar curvature of this geometry diverges at points where the heat capacity at constant angular momentum and charge diverges. According to the Davies classification of phase transitions in Black holes, this implies that the scalar curvature diverges at continous phase transition points of Kerr-Newmann black holes. Some authors disagree with the Davies classification and this issue may not be settled until when black holes are better understood.

The importance of this model of critical phenomena is that it seems to indicate that there is some physics that can be learnt from this formulation. It has been shown that the scaling behaviour of the scalar curvature and heat capacities in the extremal limit of kerr-Newmann black holes is equivalent to that for a 2-d fermi gas [4]. Although no microscopic connection has been made, future work in this area should be directed at understanding how this correspondences come about. This may lead to insight on a correct microscopic theory of black holes. No experimental results have been discussed since we are still in the every early stages of understanding black holes and there is a limited availability of experimental data.

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