Phase transitions in crowded behaviour

Thomas F. Fehm

May 13, 2010

Abstract

In this paper, I want to review theoretical models of two social phenomena: the dynamics of an applauding audience and the collective motion of people in a stadium forming a La Ola wave. In particular, I want to stress how phase transitions occur in these theoretical models and how these transitions emerge macroscopically.

1 Introduction

The rapid development of computational power during the last decades made it possible to study accumulated data of collective phenomena. Hereby, it turned out that the tools and concepts used in statistical physics can be applied very successfully to study these sort of systems: The collective motion of a large number of individuals interacting with each other gives rise to a macroscopic observable ordered patterns. However, in this paper I want to focus on collective phenomena that arise in social systems. In section 2 I want to discuss the research that has been done by Farkas et al. and Farkas and Vicsek [1, 2]. In their work, they introduced a model that was able to reproduce the dynamics of a Mexican waves, known from sport events in football stadiums around the world. In particular I will stress how the occurence of such a wave depends on the number of initiators and how it is decided in which direction the wave will propagate. In section 3 i want to discuss how a applausing audience which may form a synchoniced state. Since the underlying theoretical model - the Kuramoto model - that is used to describe the synchonication can be treated analytically, I will discuss the model itself in the beginning and derive what condition enables the synchronication. However, the application of statistical physics in the field of social dynamics is very wide and a lot of interesting research has been done during the last years. An excellent overview is given by a paper published by Castellano, Fortunato, Santo and Loreto [3].

2 Mexican waves

In this section I want to discuss the dynamics of La Ola Waves known from big sporting events. In particular, I want to stress the question how the crowd is able to make the collective decision in which direction the La Ola wave should travel and how this decision is affected by the consideration of short and long range interactions. A La Ola wave is created by spectators in a stadia when they stand up rise their arms and successively sit down when their neighbors start with the same sequence of motion. This system, consisting of individuals that synchronize their behavior, is a nice example of crowded behavior and has been studied by Farkas et al. [1] and Farkas and Vicsek [2]. In their work they proposed a model that was able to reproduce main properties of a La Ola wave like form, size, velocity, stability and gives a possible explanation how the propagating direction of the wave is decided. Farkas and Vicsek based their model on extensive analysis of recorded videos showing waves in stadia with over 50.000 participants and a online survey with 75 participants. The main conclusions they draw from this analysis were: (1) The wave is triggered by a few dozen people standing up simultaneously, (2) it is usually one wave that goes in clockwise direction and (3) spectators are interacting on local as well as on a global scale. Inspired by this experimental results they concluded that the wave is influenced by short- as well as long range interactions. Furthermore, a very short time after the wave is initiated, the crowd instantaneously decides the traveling direction of the wave. Thereby, the clockwise direction is somewhat preferred giving rise to left-right asymmetry.

2.1 Modeling the wave

Based on the conclusions mentioned previously they developed a model[1] where spectators in the stadium are represented by particles on $L_x \times L_y = 400 \times 80 = 32.000$. Furthermore, periodic boundary conditions are applied in x direction. However, spectators are considered as particles who can appear in three different states. These states correspond to different motion sequences a spectator is proceeding while participating in a La ola wave. In total, Farkas and Vicsek differentiated between three different particle states: the resting, the active and the refracting state. At the beginning of the simulation every particle on the lattice in initialized in the resting state. At every time step t the total activation effect W_i on every resting particle i is calculated:

$$W_i = G_i \sum_{\substack{j \neq i \\ j_{active}}} w_{j \to i}(\vec{r}_{ij}) \tag{1}$$

W_i is a measure how strongly the ith particle is affected by the active particles in its neighborhood. Only particles in the active state influence other particles and only particles in the resting state can be influenced. If W_i exceeds the ith particle's individual activation threshold C_i the particle changes to the active state with probability $0 < p < 1$ in the next time step $t + \Delta t$ and steps deterministically through the n_A time steps of the active state. Once finished it changes to the refracting state and again steps through another n_R time steps deterministically before it returns to the resting state where it may become active again. That particles are not activated deterministically once C_i is exceeded, takes into

[1]Actually, their model is based on a model that was studied earlier by Greenberg and Hastings [4].

account that people react differently to environmental influences. Farkas and Vicsek set $C_i = C$ for every particle but introduced another stochastic parameter to take account for individual properties of the spectators. However, two different types of interaction were tested.

2.1.1 Local interaction

In the local version of the model the long range interaction term is set to $G_i = 1$ for every particle and an isotropic, exponentially decaying interaction with characteristic length scale R is introduced:

$$w_{j \to i} = K_i^{-1} \, e^{-\|\vec{r}_{ij}\|/R} \tag{2}$$

where $K_i = \sum_{m \neq i} e^{-\|\vec{r}_{im}\|/R}$ is a normalization constant. The design of the local interaction term was motivated through the idea that only individuals in the close neighborhood influence each other. However, no direction is prefered in this interaction.

2.1.2 Local & global interaction

In the global version of the model the average distance x between the active and the ith particle was calculated and weighted with an exponential factor:

$$\bar{x}_i = \frac{\sum \Delta x_{ij} e^{-\Delta x_{ij}/X}}{\sum e^{\Delta x_{ij}/X}} \tag{3}$$

The sum runs only over active particles and the Δx_{ij} is the shorter of the two possible distances allowed by the periodic boundary conditions. X is the characteristic lenght scale of the long range interaction and $R \ll X$. With $\bar{v}_i$ being the derivative of $\bar{x}_i$ the global interaction term G_i becomes:

$$G_i(\bar{v}_i) = \begin{cases} 1, & \text{if } \bar{v}_i < 0, \\ e^{-S\bar{v}_i} & \text{if } \bar{v}_i \geq 0 \end{cases} \tag{4}$$

were the static parameter S was introduced. The velocity $\bar{v}_i$ is positive as seen from the ith particle if the active region is moving away and negative if the active region is approaching. Note that $S \to 0$ gives the local version of the model $G_i = 1$. S is a parameter that governs the long range interaction of the system. The functional form of this interaction is inspired by the idea that spectators are more influenced by a wave that approaches them.

2.2 Simulation results

A wave was triggered by moving a group of particles at $(L_x/2, L_y/2)$ with radius $\rho = 3$ to the active state [2]. In simulations where only short range interactions were present $S = 0$, two symmetric waves propagating in opposite directions occur. This result did not change as long as $S \approx 1$ (see figure 3). However, if S becomes larger, it happens that the symmetry between the two propagating waves is broken. Soon after initiation, one of the wave is selected and the other waves stops propagating [2]. The symmetric solutions becomes unstable if long range interaction is present. However, there is no left-right asymmetry. Both directions are selected equally often [2]. To study the influence of S on the symmetry of the solutions in more detail, a order parameter is introduced: During simulation, for each particle along the line $y = L_y/2$ the time of first activation is saved as a function of the coordinate x. The survival time $t_s t_s$ is no defined as the time *”below which the first activation times showed an increasing function when moving away from the initiating spot left and right”* [2]. This is indeed an order parameter: An asymmetric wave has $t_s = 0$. An stable solution has a constant survival time, namely the time that is needed till the waves meet at the lattice. Figure 4 shows the probability distribution of t_s for different values of S. Clearly, one can see the distinct peaks. This distribution is analogous to the distribution of the order parameter of a system close a a transition point where the system performs a discontinuous phase transitions [2]. The system changes from the symmetric solution with two waves and large t_s to the asymmetric solution with only one wave and small t_s. The reason why the transition is not very sharply is that the number of initiating particles is finite and therefore the data is influenced by finite scaling effects. The inset of figure 4 supports this conclusion. The average survival time $< t_s >$ is plotted as a function of S. The data collapses as expected in the vincity of the critical point S_c. The transition becomes sharper as X increases. The critical value of S scales like $S_c \propto X^{-1/2}$ [2]. Farkas et al. showed that the long range interaction play an important part in how the travelling direction is decided. It seems an reasonable assumption to design the interaction like they did, namely that an approaching wave should influnce more than a passing wave. The model is able to produce an stable asymmetrical solution but still, there are two waves in the beginning. In real Mexican waves the direction of the wave is decided much faster and consequently only one wave can be observed right from the beginning. In addition, the ratio of waves to the left and to the right was one. This seems unsatisfying in the context of the the experimental resulsts mentioned in the beginning. Furthermore, the triggering process, setting a number of particles uncorrelatetly to the active state is very unrealistc. However, Farkas et al. modified the global model assuming that people react asymmetrically to events to their left or to their right. If this is true, it should also affect the decision process of the propagation direction. The local interaction term was modified by anisotropic factor:

$$w_{j\to i} = \frac{e^{-\|\vec{r}_{ij}\|/R}\,((1-\delta)+\delta\cos(\pi-\phi))}{K_i} \quad (5)$$

where ϕ is the angle between $\vec{r}_{ij}$ and the ith particle local reference frame. $\phi = 0$ if $\vec{r}_{ij}$ points to the left and the clockwise direction is positive. Depending on the value of δ, the left-right symmetry can now be broken. If δ increases the probability of a right moving wave increases and will be completely dominant over the left moving wave (see figure: 5). However, the scaling of the critical value is not affected $S_c \propto X^{-1/2}$ [2].

Besides the symmetry breaking, Farkas et al. studied the influence of the size of the triggering group. Figure 8b shows the probability that a wave is observed depending on the size of the triggering group and the activation threshold C. In this simulation they used a slightly different decision rule when a particle becomes active. Instead of a general activation threshold C and a activation probability, every particle has its own activation threshold C_i chosen randomly out of $[c-\Delta c\,;\, c+\Delta c]$ and it is actived deterministically once $W_i > C_i$. The results show that the probability is sharply changing once a critical parameter set is exceeded. This suggests that triggering a Mexican wave requires a critical amount of initiators.

3 The applauding audience

It is common to applaud after a good theater show. People show their appreciation of the performance and clap their hands collectively. However, from time to time it happens that the applause synchronizes resulting in a rhythmic clapping noise. This phenomena of synchronization can be studied in the context of the Kuramoto model. Kuramoto's model [5, 6] is perfectly suited to understand synchronization processes both quantitatively and qualitatively. An nice example the Kuramoto model can be applied to is an applauding audience that synchronizes its clapping rythm [7, 8]. A great inherent feature of the Kuramoto model is that it is possible to analytically derive a condition that the oscillators may synchronize. Since this result is so enlightening, I would like to present a deviation of this cirteria first. Subsequently, I will explain how this criteria can be applied to the applauding audience.

3.1 The Kuramoto model

Great effort was put into the development of a mathematical model to describe the phenomenon of collective synchronization of an enormous number of interacting oscillators. Examples for synchronization processes can be found everywhere in nature and subsequently cached the interest of scientists. A beautiful introduction to this topic with more interesting examples of synchronization process and how the occur is given by [9]. In 1967, Winfree [10] suggested a model where every oscillator is coupled to a mean frequency that is generated by the hole population:

$$\dot{\theta}_i = w_i + \left(\sum_{j=1}^{N} X(\theta_j) \right) Z(\theta_i) \tag{6}$$

for $i = 1, \ldots, N$. $\theta_i \in [0, 2\pi[$ is the phase of the ith oscillator in the laboratory frame and w_i its natural frequency. Each oscillator is affected by the population influence described by $X_(\theta_j)$. The answer of the ith oscillator to this influence is describes by $Z(\theta_i)$. Using computer simulations, Winfree made an important observation: If the spread of the natural frequencies $w_i$is large compared to the coupling strength, the oscillators do not synchronize. However, if the spread decreases there is a critical point, once crossed a fraction of rotators suddenly synchronize oscillating all with the same frequency. This fraction increases if the spread decreases further [10]. Based on Winfree's, model Kuramoto published a paper in 1975 [5, 6] where he showed that for any system of weakly coupled oscillators with limit cycle, the long term dynamics of the system can be described with only one function depending on the phase difference of the considered oscillators: $X(\theta_j)Z(\theta_i) = \Gamma_{ij}(\theta_j - \theta_i)$. Furthermore, by simplifying $\Gamma_{ij}(\theta_j - \theta_i) = K/N \sin(\theta_j - \theta_i)$ with $K \geq 0$, he was able to solve the model analytically giving a mathematical condition when synchronization occurs. In this simplification, the interaction weight and the functional form is the same for every interaction pair. The $1/N$ factor ensures that the thermodynamic limit ($N \to \infty$) exists. The is called the Kuramoto model:

$$\dot{\theta}_i = w_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) \tag{7}$$

and $i = 1, \ldots, N$ [5]. He further assumed that the frequencies are distributed according to a density function $g(\omega)$. For simplicity this function shall be unimodal and symmetric about a mean frequency Ω: $g(\Omega + \omega) = g(\Omega - \omega)$. A measure of how synchronized the oscillators are is given by [5]:

$$re^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j} \tag{8}$$

ψ can be thought of as the average phase of all oscillators and r is a meassure of phase coher-

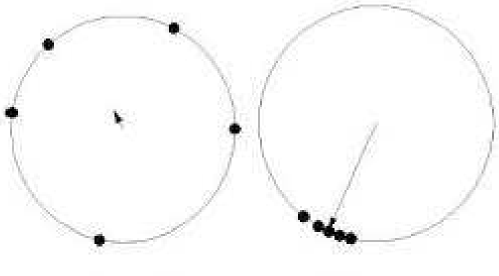

Figure 1: Geometric interpretation of order parameter, equation 8. The black dots correspond to the phases θ_j. The center is given by the complex number $re^{i\psi}$. If the system is completely disordered (like in case (a)) r is approximately zero. If every oscillator is in the same phase $r \approx 1$ since all vectors are pointing in the same direction (case (b)). From [11].

ence of the system. If every oscillator is exactly in the same phase $\theta_j = \theta$ the right hand side of equation 8 simplifies to $Ne^{i\theta}$ and $r = 1$ with $\psi = \theta$. On the other hand, if the system is completely disordered, meaning that the θ_j are distributed uniformly over the interval $\theta_i \in [0, 2\pi[$ the sum equals zero and therefore $r = 0$. To make this idea more clear one can imagine the state of every oscillator as a vector pointing from the origin to the corresponding phase on a unit circle (see figure: 1). The mean field character of Kuramotos model becomes obvious if one rewrites equation 7 in terms of r and ψ [6]. Multiplying both sides of equation 8 by $e^{-i\theta_i}$, subtracting its own complex conjugate and dividing both sides by $2i$ yields:

$$r\sin(\psi - \theta_i) = \frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) \tag{9}$$

which is exactly equation 7. Thus it follows:

$$\dot{\theta}_i = w_i + Kr\sin(\psi - \theta_i) \tag{10}$$

Every oscillator is influenced only by the mean field parameters r and ψ. Suppose for a second that r equals zero, so there is no synchronization. Equation 10 tells us that every oscillator oscillates with his own natural frequency. If r increases the coupling to the mean phase ψ becomes stronger pulling the oscillator towards the mean frequency. Instead of solving equation 10 in general, Kuramoto was now looking for steady solutions where the system is already in the synchronized state and any non equilibrium behavior died off. So, $r(t)$ is constant and $\psi(t)$ rotates uniformly at a frequency Ω. If we transform the system into the frame which is rotating at this frequency Ω: $\theta_i \to \theta_i + \Omega t$ (note that this implies $g(\omega) = g(-\omega)$), ψ is a arbitrary constant and therefore can be set to zero[2]. In the new frame, equation 10 reads[3]:

$$\dot{\theta}_i = w_i - Kr\sin(\theta_i) \tag{11}$$

This equation determines the steady solutions in the rotational frame. Oscillators with $|w_i| \leq Kr$ will have the steady solution $\dot{\theta}_i = 0$ and so $\omega_i = Kr\sin(\theta_i)$ [12]. They freeze at a particular phase $\theta_i = arcsin(\omega_i/Kr)$ in the new frame. These oscillators are called locked since they are locked the the frequency Ω in the original frame. Oscillators with $|w_i| \geq Kr$ will not able to frequency lock $\dot{\theta}_i \neq 0$. To ensure that r and ψ will still be constant even with this drifting oscillators Kuramoto assumed that the drifting oscillators from a stationary distribution on the circle. So, in areas of the unite circle where less oscillators are,

[2]This means setting $\psi(t) = 0$ if $t = 0$.
[3]Naturally, $\omega_i \to \omega_i + \Omega$ in the new frame.

they have to move faster than in areas were the density is high. Let $\rho(\theta, \omega)d\theta$ be the fraction of oscillators with frequency ω that lie between θ and $\theta + d\theta$. Stationarity implies then:

$$\rho(\theta, \omega) = \frac{C}{|\dot{\theta}|} = \frac{C}{|w - Kr\sin(\theta)|} \tag{12}$$

with C defined by $\int_0^{2\pi} \rho d\theta = 1$.

Since we are looking for steady solutions in the rotational frame only, $\psi = 0$ and equation 8 becomes

$$r = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} \tag{13}$$

with θ_j obtaining equation 12. The sum can be split into the locked and drifting oscillators $r = r_{locked} + r_{drift}$. The locked oscillators contribute:

$$\begin{aligned} r_{\text{locked}} &= \frac{1}{N}\sum_{\text{locked}} cos(\theta(\omega_i)) + i\frac{w_i}{Kr} && (14)\\ &= \frac{1}{N}\sum_{\text{locked}} cos(\theta(\omega)) && (15)\\ &= \int_{|w|\leq Kr} cos(\theta(\omega))g(\omega)d\omega && (16) \end{aligned}$$

in the limit of $N \to \infty$ and $w = Krsin(\theta)$. Recall, that the natural frequencies are assumed to follow a symmetric unimodal distribution $g(\omega)$ about zero in the rotational frame. The contribution of the drifting oscillators is:

$$r_{\text{drift}} = \int_0^{2\pi}\int_{|w|\geq Kr} e^{i\theta} g(\omega)\rho(\theta,\omega)d\omega d\theta = 0 \tag{17}$$

since equation 12 obtains: $\rho(\theta + \pi, -\omega) = \rho(\theta, \omega)$, equation 12 and $g(\omega) = g(-\omega)$. Changing the integration variable in equation 16, finally yields to a self consistent equation for r:

$$r = Kr\int_0^{\pi} cos^2(\theta(\omega))g(Krsin(\theta))d\theta \tag{18}$$

Note that $|w| \leq Kr$ implies $\theta \in [0, \pi]$. Obviously $r = 0$ is always a solution for any value of K. A second solution can be obtained by letting $r \to 0^+$ in equation 18. This gives us the critical value K when oscillators start to synchronize collectively.

$$K_c = \frac{2}{\pi g(0)} \tag{19}$$

Kuramoto showed in [5, 6] that in the special case of a Lorentzian density $g(\omega) = \frac{\gamma/\pi}{\gamma^2+\omega^2}$ equation 18 can be integrated analytically yielding:

$$r = \sqrt{1 - \frac{K}{K_c}} \tag{20}$$

for all $K \geq K_c$. This is very important result. The system of oscillators can not synchronize unless the coupling strength exceeds a certain critical threshold. Once this threshold is exceeded, the system performs a phase transition. The population is divided into oscillators that freeze at a particular frequency Ω and oscillators that drift acceleratingly and decceleratingly around the unite circle giving rise to partial order in the system $r \neq 1$. The fraction of drifting oscillators becomes smaller if K increases further. In the limit $K \to \infty$ all oscillators are frequency locked and we have perfect coherence $r = 1$ (see figure 2). Of course, the sinusoidal simplification that was made is not true in general and might even not be a good approximation in most cases. However, it supports us with the possibility to derive a formula that gives us an understanding how synchronization can arise. The Kuramoto model is still discussed controversely and at lots of questions are still unanswered. For example it is still not proven that the steady solution $r \neq 0$ is stable [12] nor is it proven that Kuramotos approach, that was presented

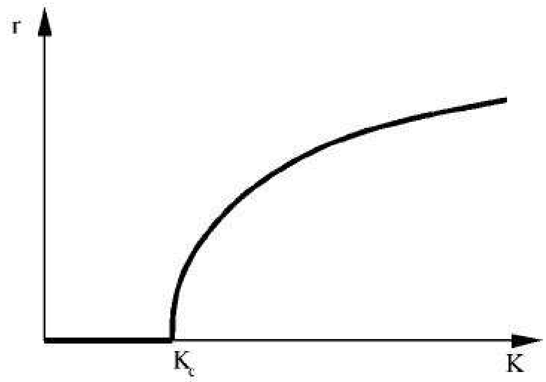

Figure 2: Bifurcation in the Kuramoto model for unimodal $g(\omega)$. From [13].

here, is rigeriously valid in the thermodynamic limit [12]. However, the Kuramoto model was applied successfully to a wide range of problems and lots of highly interesting modifications like stochastical perturbation have been developed [13, 14]. A beautiful and readable introduction to the Kuramoto Model is given by Strogatz [12].

3.2 Experimental findings and conclusion

In 2000 Néda, Ravasz and Vicsek published two papers [7, 8] where they showed that an applauding audience can be analyzed as a population of oscillators within the framework of the Kuramoto model. Their conclusion is based on two experiments [7, 8]. In the first experiment they recorded the applause after theater performances using microphones that were hanging from the ceiling as well as placed in the neighborhood of randomly selected individuals. They define and calculate an oder parameter q_{exp} [7], similar to the one in equation 8 as well as the noise intensity of the signal, averaged over 3 seconds. A typical result is shown in figure 7. 7a shows the global short time average of the recorded signal after digital preprocessing. Figure 7b shows the local signal that was recorded next to an individual. It can be observed that at about $t = 12$ seconds, the signal becomes periodic. The calculated order parameter in 7d increases. Moreover, the long time (3 seconds) noise intensity signal, 7c, decreases when the clapping becomes more synchronized having its minimum where the order parameter is maximal. Figure 7d shows the calculated clapping frequency of an individual as a function of time. One can see that the individual changes its clapping frequency to approximately half the frequency while the order parameter is rising. Their conclusion was that there are two different types of clapping. A type I clapping when individuals applaud independently and a type II clapping that appears when the audience is synchronized. In a second experiment, they recorded the clapping of 73 high school students that were asked to clap as they would right after a good theater performance (type I) and as they would during synchronized clapping (type II). This experiment was repeated with one student during one week for 100 times. The results of this experiment were: (1) The frequency distribution of type I clapping is larger than of type II clapping[4]: $D^I/D^{II} \approx 2.5$ (figure 7f). A similar behavior can be observed from the 100 measurements on one individual (figure 7g). (2) The ratio of the peak-frequencies of type I to type II is approximately $2:1$ (figure: 6) as it can be observed in figure 7f,g. Néda, Ravasz and Vicsek concluded, that people lower their clapping frequency to a value that is half as large as before in order to achieve synchronization. This is exactly what one would have expected considering the audience as a popula-

[4] A Gaussian was fitted through the data to obtain the dispersions [7]

tion of oscillators in the Kuramoto model with θ_i being the clapping phase of the individual and ω_i its natural clapping frequency. Solving equation 19 in the case of a Gaussian frequency distribution with dispersion D yields: $K_c = \sqrt{\frac{2}{\pi^3}}D$. Synchronization is only possible if the coupling exceeds the critical value K_c. By changing from type I to type II clapping the audience reduces the dispersion and therefore lowers the critical coupling strength until synchronization is possible. Of course, this does not explain why people try to synchronize their clapping. However, figure 7d indicates that the synchronization is lost at about 20 seconds. Néda, Ravasz and Vicsek explained this due to the fact that the average noise decreases during synchronized clapping. The audience might feel that this is not satisfying and therefore tries to increase the noise level by clapping faster. The frequency distribution increases again and so does the critical coupling strength. The coupling among the people may become smaller than the critical value and synchronization is lost. A quite good example for an audience clapping rythmicly can be found here (listen carefully at about second 8). However, the model is not really realistic since one would assume that spectators are driven only by the desire to increase the global noise level. In a more recent paper [15] Néda et. al introduced a new model where they treat the spectators as a two mode stochastic oscillator which are only driven by exactly this goal.

4 Conclusion

The two models that have been introduced seem to have a lot of potential. Especially the Kuramoto model seems to be very interesting in the context of understanding synchronization in in general. Using comparatively easy mathematical tools, a strong criteria could be derived giving a deep intuitive understanding of how synchronization occur. Farkas et al. also presented a very interesting model for crowd behavior that might not only be used to describe Mexican waves but also in general cases, when the dynamic of social group is heavily influenced by a little fraction. However, the powerful methods of statistical physics will contribute to the understanding of system with a large amount of interacting members and will hopefully lead to a deeper understanding of the complex patterns of social interactions.

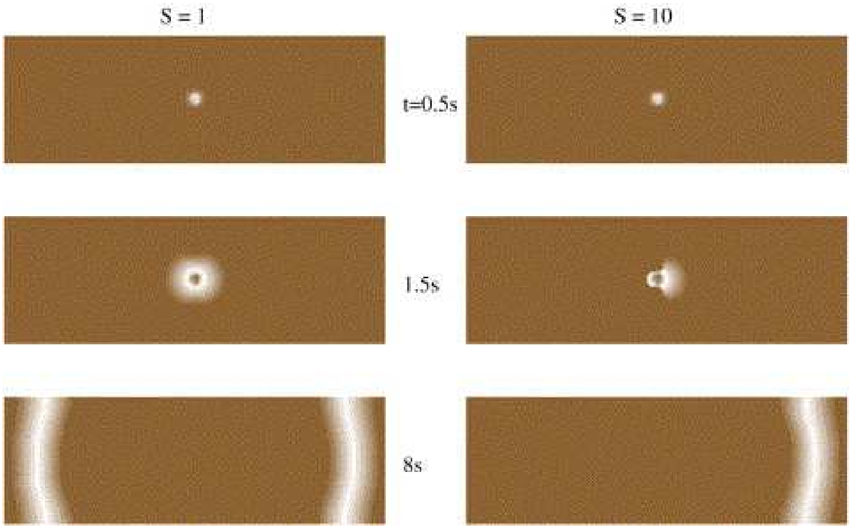

Figure 3: Spontaneous symmetry breaking in the Mexican wave simulation. The increasing brightness indicates the different states of motion of the spectator. White means standing with hands up. From [2].

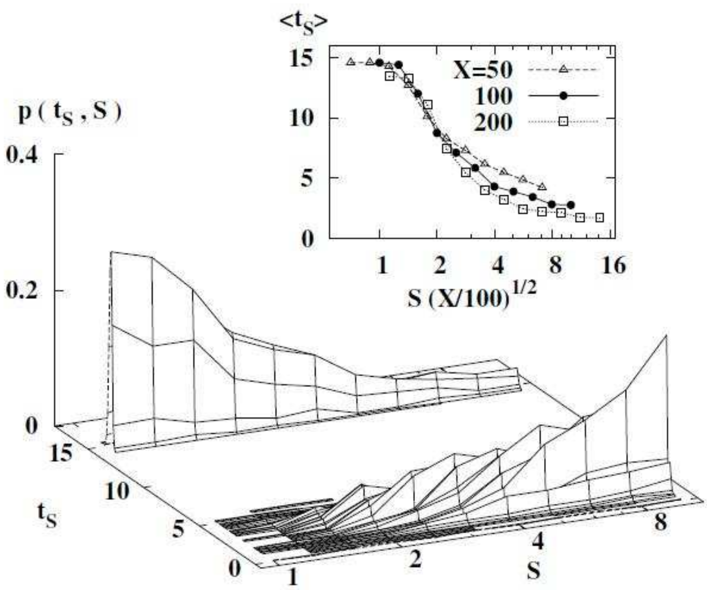

Figure 4: Transition between symmetric and antisymmetric solution. From [2].

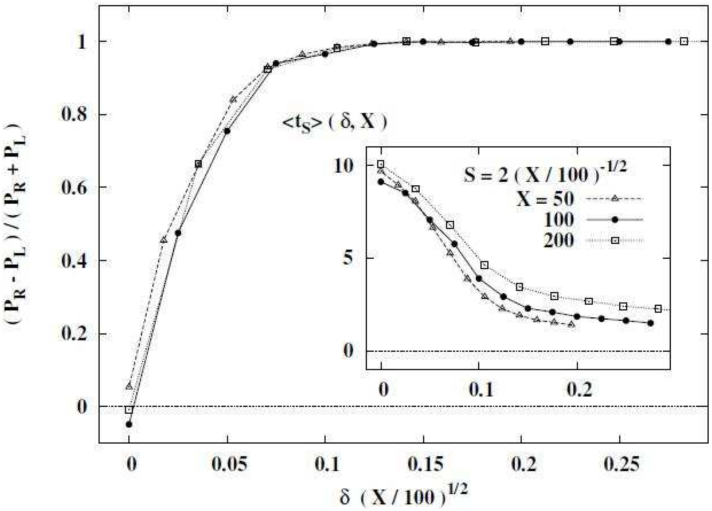

Figure 5: Probability distribution for propagating direction as a function of δ. From [2].

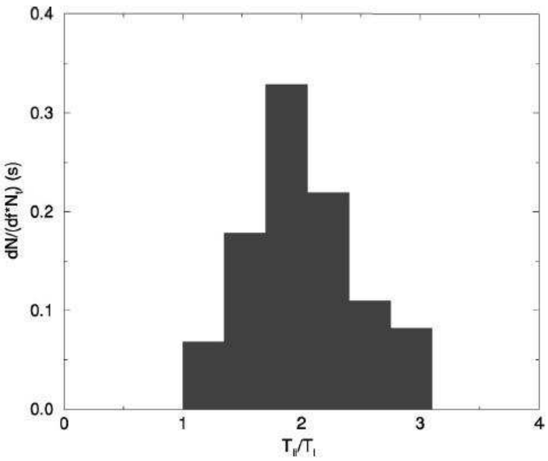

Figure 6: Normalized distribution for the ratio of frequencies of type I and type II clapping. From [7]

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