Explosive phase transitions in percolation processes

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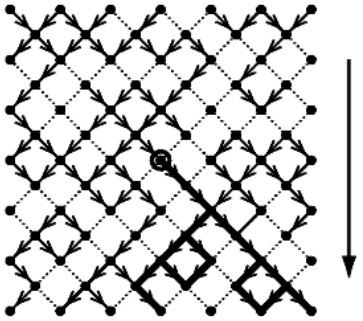
Abstract

Percolation processes are well studied in physics. In theoretical physics, directed percolation (DP) is a representative of a well-known universality class of continuous phase transitions [1]. DP has been used to model a variety of phenomena including turbulence, liquids percolating through porous media, epidemics and forest fires [2]. In the Erdös-Rényi model, it is known that the order parameter (size of largest connected component) undergoes a continuous phase transition beyond a critical percolation treshold. However, recent numerical experiments indicate that an Erdös-Rényi model with a modified percolation rule can have an "explosive" phase transition in which the order parameter undergoes a discontinuous jump [3, 4, 5]. Another model of a random network of integrate-and-fire neurons shows a similar transition [8].

1 Introduction

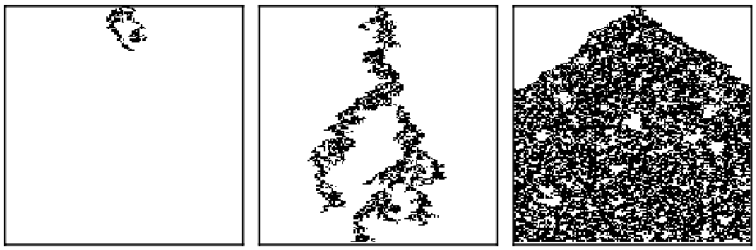
In this paper we will describe a number of statistical mechanical models. Unifying feature of all of these models is that they are percolation-like. That is, the underlying graph structure has edges that are either open or closed with some probability. Consider the following figure (adapted from [1]). The diagram illustrates directed bond percolation on a diagonal lattice of nodes. Each two adjacent nodes in the lattice are connected by bond that is open with probability p and closed with probability $1 - p$.



directed bond percolation

Imagine that a fluid is allowed to flow through the lattice. It is driven by gravity, hence it can only flow down-left or down-right (indicated by direction of arrows), and only through open bonds (indicated by solid lines). Then, a size of a percolating cluster is the number of nodes that are reached by the fluid starting from the nucleation point.

Evidently, if the probability p that any bond is open is large, then the percolating cluster will be large (on the order of the size of the system, that is number of nodes). If p is small, we expect that the percolating cluster will be small, and will not scale with the size of the system. Hence, there is some threshold p_c above which the fluid will percolate to any depth from the nucleation point. This is nicely illustrated by the following diagram (adapted from [1]).



This transition (along p) is continuous and is characterized by a number of universal critical exponents. Contact processes, forest fire and epidemic models have all been found to belong to the directed percolation universality class. However, the directed percolation exponents have only recently been experimentally measured (i.e. non-numerically) in an interesting experiment on turbulent liquid crystals [6, 7].

2 Introduction to the Erdös-Rényi model

Whereas directed percolation is a spatially extended process, in this section we will consider the Erdös-Rényi model, which is an infinite-dimensional percolation process, i.e. a mean field model. However, the explosive transitions that are the topic of this paper are not specific to mean field models. We will see later that they also occur in percolation models on spatially extended lattices.

In the Erdös-Rényi model, one starts with n vertices that are initially all isolated. Then, one begins to add edges to the vertices by picking any two random nodes and connecting them. Now, two nodes are said to be in the same component if there is a path of edges that connects them. We will call the number of nodes in the largest such component C.

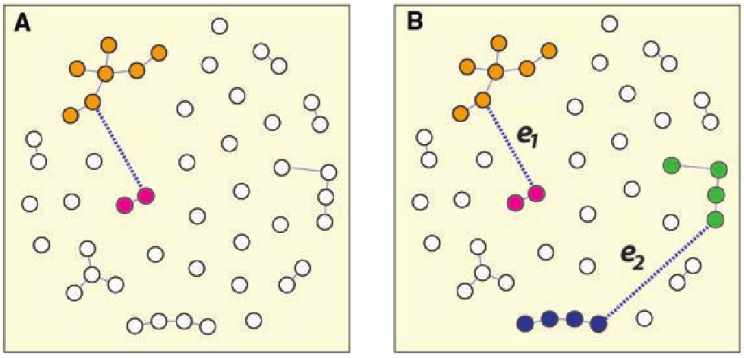
Evidently, as more edges are added, C will grow, until at some point all nodes are a part of one giant component and $C = n$. But in what way does C scale with the number of edges e? We will define r via $e = rn$. This defines r as the average degree of the nodes in the graph (i.e. the average number of neighbours each node has in the graph.) For specific n and e the number of possible graphs is extremely large, and the problem of characterizing such graphs is mathematically intractable. One of the brilliant insights of Erdös and Rényi was to realize that the random graphs in the limit $n \to \infty$ are well-defined [8]. For such graphs, it is possible to calculate a variety of measures,

among others including the scaling of C. Erdös and Rényi found that in the standard random graph model, when $r < 0.5$, $C \sim \log n$. But when $r > 0.5$, $C \sim (4r - 2)n$. That is, above the critical value $r_c = 0.5$, the size of the largest connected component scales linearly with n.

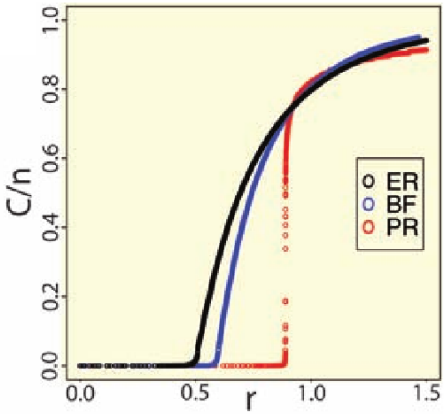
3 Achlioptas process

The idea behind the Achlioptas process [4] is to modify the rule for generating Erdös-Rényi random graphs. Instead of adding random edges, in the Aclioptas process one picks two edges at random, and then uses a rule to select one or the other. The selected edge is then added to the graph whereas the other edge is returned to the pool of missing edges. The process then continues.

One of the selection rules Achlioptas *et al.* [4] considered was the product rule (PR). In PR, one chooses the edge that minimizes the product of the sizes of components on each side of the edge. As an example, consider the following diagram (adapted from [4]). In A, the standard Erdös-Rényi model is illustrated, where a single random edge is picked and added to the graph. In B, two edges, e_1 and e_2 are picked at random. Then e_1 is chosen because its product $(2 \times 7 = 14)$ is smaller than that of e_2 $(4 \times 4 = 16)$.



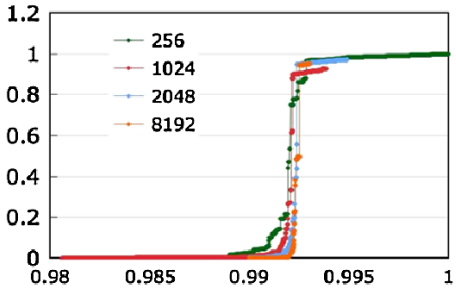
The interesting feature of adding such a nonrandom selection rule to the Erdös-Rényi model is that the percolation threshold can be reduced or increased (i.e. percolation can be either accelerated or suppresed). In the case of PR, the percolation threshold is increased. Consider the following diagram (adapted from [4]) that illustrates the numerical results for the Erdös-Rényi model (ER), bounded size rule (BF) and the product rule (PR). BF is another Achlioptas rule that can increase the percolation threshold.



These numerical results were computed for $n = 512,000$. Whereas the percolation threshold is increased for both BF and PR, compared to ER, the interesting result here is that it seems like the transition for PR has a discontinuous jump.

The figure above is only enough to motivate the idea that the PR percolation transition is discontinuous. Achlioptas *et al.* use an interesting method to verify that the transition indeed is discontinuous. They numerically measure the time step t_0 at which C first becomes larger than $n^{1/2}$ and t_1 at which C first becomes larger than $0.5n$. A time step corresponds to a value of r via $t = rn$ (since the number of edges at time step t is t). For the ER process, and other continuous percolation transitions, it is found that $\Delta \equiv t_1 - t$ is extensive, i.e. it scales with n. On the other hand, for the PR model they find that $\Delta \sim n^{2/3}$. Since Δ is sublinear in n, this is numerical evidence that in the thermodynamic limit $n \to \infty$, the percolation transition indeed is discontinuous.

Discontinuous transitions aren't unique to mean-field systems. Ziff has investigated the PR model on a 2D lattice and he finds that it also exhibits a discontinuous transition. The figure below (adapted from [3]) illustrates the numerical results on lattices of linear dimension 256 through 8192.

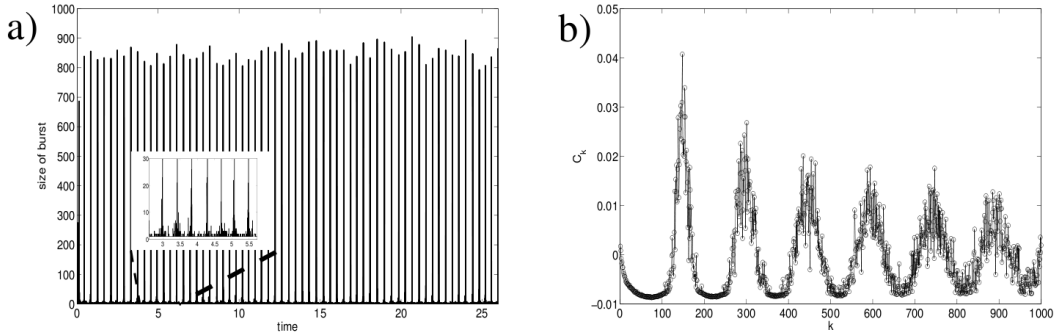


4 Explosive transitions in a neural network model

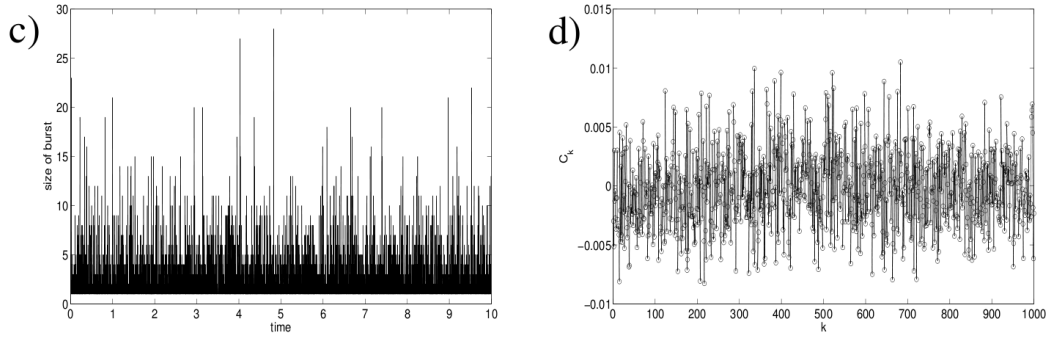
A more interesting model for a random network of discrete integrate-and-fire neurons has been investigated by DeVille *et al.* [8]. Same sort of an explosive transition can occur in this model, but this model has a more physical motivation. In this section we will briefly describe the model given in [8] and explosive transitions that occur in that model.

In this model there are n discrete integrate-and-fire neurons that are all attached to each other with excitory synapses. These neurons can be in K different discrete levels of excitations, $0, \ldots, K-1$. If a neuron is promoted beyond level $K-1$, it itself fires and promotes all other neurons in the network with probability p. A neuron firing can promote a cascade of other neurons firing (a burst). Once the burst is complete, all the neurons that fired are reset to level 0. Each time step a single neuron is promoted, and this may lead to a cascade of other neurons firing.

In this model the size of the largest component C is replaced by the size of a burst of neurons. When p is large, extensive and synchronized bursts of neurons occur. The figure below (adapted from [8]) shows (a) the time series for C, and (b) the autocorrelation of the time series for C, indicating the synchronization. This figure is for a numerical simulation on 1000 neurons.

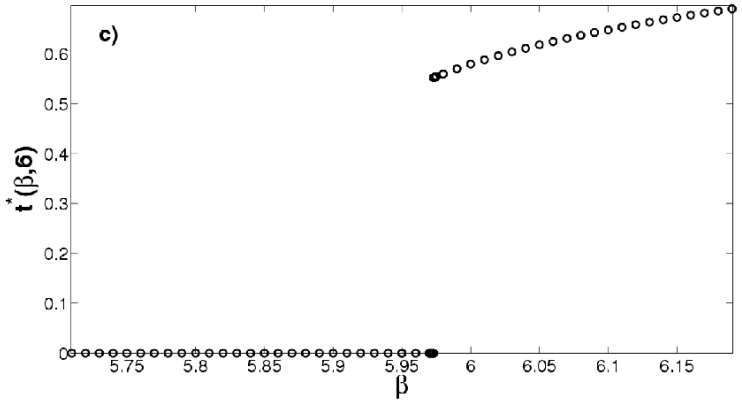


On the other hand, when p is small, the time series is qualitatively different. The figure below shows the dynamics for small p. In (c) we see that the sizes of bursts are small and (d) their autocorrelation doesn't show any synchronization.



Another interesting feature of this model is that there exists a parameter range of p in which these two states (synchrony and asynchrony) are metastable and the system can make transitions between the two states. The numerical results indicate that the times spent in each of the states are distributed exponentially indicating a memory-less transition process between the two states.

Evidently, when $K = 1$ this model is equivalent to the Erdös-Rényi random graph construction process. Interestingly, the order of the transition between the microscopic and macroscopic bursts seemingly depends on K. DeVille *et al.* find numerical evidence that when $K \geq 5$ the transition in fact becomes discontinuous. The following figure (adapted from [8]) shows the numerical results for the fraction of neurons participating in a burst as function of the synapse probability ($\beta \sim p$) for $K = 6$.



5 Conclusion

The numerical evidence is strong that the percolation processes can have a discontinuous transition in some models. However, a mathematical proof is still missing even for the simplest model (PR). This is because the combinatorics of the PR process is a great deal more complicated than that of the Erdös-Rényi model.

However, it is unclear if the language of phase transitions in physics can be used to characterize these "explosive" transitions. I believe the key difficulties in calling these transitions first-order are: (1) the "order parameter" (size of largest cluster, or burst) is non-local, and (2) it is difficult to construct a free energy as function of such order parameter.

References

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