Limit Cycles in the Renormalization Group

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(Dated: May 11, 2012)

This paper will review the consequences of having limit cycles in the renormalization group. Characteristics such as discrete symmetry invariance and log periodic behavior of observables are expected in systems that exhibit limit cycles in their renormalization group. Some realizations of limit cycles will be explained. Specifically limit cycles in RG flows will be used to explain the Efimov effect.

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I. INTRODUCTION

The Renormalization group was created by Wilson[1] to explain Kadanov-Widom scaling and has had enormous success in explaining critical phenomena. Additionally, it has been successful in many other branches of physics including particle physics, where it explains the physical meaning of divergences in field theories. It also has a convenient explanation for universality of certain phenomena. Almost all these examples involve a fixed point in the Renormalization Group (RG). Fixed points of the RG transform allow one to find critical exponents and explain the power law scaling laws found near criticality. Near fixed points one sees a scale invariance of the system. In this paper I will discuss an alternate possibility, chiefly the consequences of having a limit cycle in the RG flows.

II. THE RENORMALIZATION GROUP

The renormalization group consists of two steps. The first step is a coarse graining step. In this step one integrates out of short-distance degrees of freedom of distance less than l. The second step is to rescale distance back to their original length scale. Thus the degrees of freedom look like the original system but now have a new effective Hamiltonian. One can write the Hamiltonian as:

$$H_l = \sum_n K_{n,l} O_n$$

where $K_{n,l}$ are the coupling constants at scale l and O_n are the local operators. The renormalization group transform defines a flow through coupling constant space. After a series of renormalizations the effective hamiltonian changes. This flow of coupling constants can be expressed as a differential equation for the coupling constants of the form:

$$\frac{d[K_\tau]}{d\tau} = B[K_\tau]$$

where $\tau = \log[l]$. The condition for a fixed point in this flow is $B[\vec{K^*}] = 0$. By linearizing around the fixed point corresponding $\vec{K^*}$ one finds eigenvectors of coupling constants and their corresponding eigenvalues. When an eigenvalue is positive the coupling constants will grow along the corresponding eigendirection. When the eigenvalue is negative the coupling constants will come back to the fixed point corresponding to an irrelevant direction.

One important consequence of RG is it allows one to understand the origin of singular behavior from a transform that is non-singular. It turns out that this is just a consequence of looking at what happens after an infinite number of iterations in the thermodynamic limit or as formulated here $\tau \to \infty$. This is analogous to the classical mechanics problem of starting a ball on some topography. After an infinite amount of time the ball will have rolled to a local min and stay there. Thus depending on the initial condition you have different basins of attraction that correspond to different local minimums. But if you only consider finite time then your final position as a function of initial position is still a continuous function and you may not have yet reached a local minimum. Thus we see we are required to take the infinite time limit to recover singular behavior. This also gives an explanation for universality, i.e. Hamiltonians in same basin all flow to the same fixed point. The corresponding fixed point eigenvalues determine critical exponents. [2]

III. NONLINEAR DYNAMICS CLASSIFICATIONS

In nonlinear dynamics there are a few common possible features of flows in phase space. In the case of RG this phase space is the space of coupling constants for all local operators of the Hamiltonian. The most common situation encountered in RG is that of a fixed point. A fixed point is a point in the space that remains constant. It is fixed under a RG transformation. Fixed points can be classified by their stability. In RG one generally uses linear stability analysis to classify the fixed point. To perform linear stability analysis one first linearizes the transformation at the fixed point to see what happens to a small perturbation away from the fixed point. If the eigenvalues are positive that eigen direction is unstable or in RG terminology "relevant". Small perturbations in a relevant direction will grow. If the eigenvalue is negative the fixed point is stable or in RG terminology "irrelevant". Finally if the fixed point has a zero eigenvalue one must go beyond linear stability analysis to find its stability. Eigendirections with eigenvalues zero are called "marginal". These are generally all that are discussed in typical RG textbooks. But other possibilities exist.

If we are considering a system with at least two coupling constants, eigenvalues can also be imaginary. Imaginary eigenvalues correspond to spirals in phase space. If the real part of the eigenvalue is positive one finds spirals going away from the fixed point, if the real part is negative one finds spirals going towards the fixed point. Finally if the eigenvalues are purely imaginary the fixed point is called a center and has a family of closed orbits surrounding the fixed point. Any small perturbation will result in becoming stuck forever on one of these closed orbits, orbiting the fixed point.

In addition to fixed points other possibilities exist for flows in phase space. One possibility is chaotic behavior. Chaos in nonlinear dynamics is characterized by high sensitivity to initial conditions. Two points started very close to one will exponentially grow apart after only a few iterations.

Another behavior is that of limit cycles. A limit cycle is an isolated closed trajectory. See the following figure for a caricature drawing of some limit cycles as compared to a center. Limit cycles are nonlinear phenomena. They require at least two coupling constants to be present (unless there are discontinuities in a single coupling constant). Limit cycles are different from centers in that they are determined by the structure of the system, while linear oscillators are determined by their initial conditions. For the rest of this paper I will consider limit cycles.

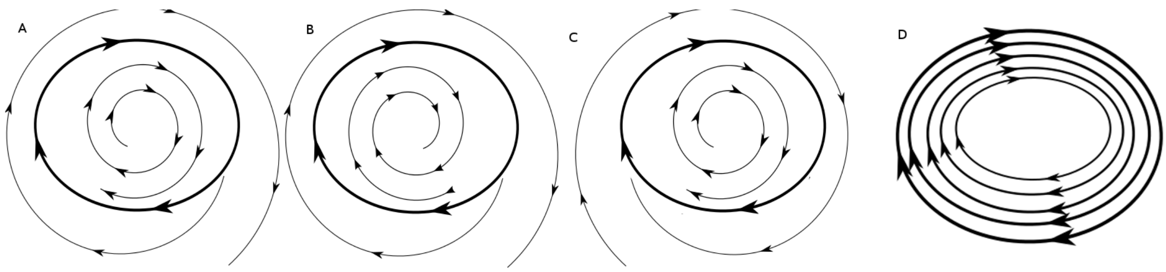

FIG. 1: Limit cycles are isolated closed trajectories in phase space as can be seen in figures A, B, and C. A: Half-Stable limit Cycle B: Unstable limit cycle C: Stable Limit Cycle D: Center

Interestingly it is possible to rule out limit cycles or closed orbits if the RG transform has a certain form. If the RG equations can be written in the form of a gradient system then closed orbits are impossible and by extension limit cycles. A gradient system is one that can be written as $\dot{x} = -\nabla V$ for a continuously differentiable potential $V(x)$. Another class of RG transforms that cannot have closed orbits are ones where it is possible to define a Liapunov function. A Liapunov function is an energy like function that decreases along trajectories. Since we are interested in looking for systems with limit cycles we must therefore exclude RG transforms that can be written as a gradient system or for which a Liaponov function can defined. [3]

IV. CONSEQUENCES OF LIMIT CYCLES

A limit cycle in RG can be described by a family of Hamiltonians $H_*(\theta)$ that is closed and can be described by an angle that runs from 0 to 2π. Also the coupling constants return to their original value after the cutoff has been changed by a factor S_0 called by Hammer and Platter the preferred scaling factor:

$$K(l) = K_*(\theta_0 + 2\pi \ln(l/l_0)/ln(S_0))$$

. We know along RG flows that correlation length decreases (by a factor of l in discrete RG transformations $\xi[K'] = \xi[K]/l$). Thus using a similar argument to the one for fixed points except using the preferred scaling factor you see that ξ must be infinite or zero. Limit cycle attractors have $\xi = 0$ and repelling limit cycles have $\xi = \infty$. Since we are interested in singular behavior we are thus interested in repellent limit cycles. To see the consequences of Limit cycles I will follow Veytsman and consider the same repellent limit cycle he does.[4] The RG transform in this case can be written in polar coordinates.

$$\frac{d\rho}{d\tau} = k(\rho - \rho_0), \frac{d\phi}{d\tau} = w(\rho)$$

with τ as the scaling parameter and ρ_0 is the radius of the limit cycle. Veytsman finds a partial differential equation for the correlation radius:

$$k(\rho - \rho_0)\frac{d\xi}{d\rho} + w(\rho)\frac{d\xi}{d\phi} = 0$$

Solving this equation for correlation radius he finds a solution:

$$\xi = (\rho - \rho_0)^{-1/k}\Xi(\phi - \int^{\rho} \frac{w(x)}{x} dx)$$

where Ξ is an arbitrary periodic function of period 2π. Thus we see a divergence of the correlation radius at ρ_0. We see far from ρ_0 that the correlation radius is smooth except possibly at a few points the Ξ function. These discontinuities would correspond to lines of phase transitions. Additionally Veytsman shows that the free energy can be given by:

$$f = (\rho - \rho_0)^{d/k}F(\phi - \int^{\rho} \frac{w(x)}{x} dx)$$

Where F is again an arbitrary periodic function of period 2π. Expanding $w(\rho)$ to leading order $w_0(\rho - \rho_0)^n$ one finds the argument to the Ξ function to be either $\phi - w_0/n(\rho - \rho)^n$ for n not equal to zero or $\phi - w_0 \ln(\rho - \rho_0)$ for $n = 0$.

Now lets find the susceptibility $\chi = f_{\rho\rho}$. In the case of n = 0 the susceptibility is bounded by the function $\chi = Constant(\rho - \rho_0)^{d/k-2}$ and the susceptibility oscillates as a function of ρ. Since the relaxation time is proportional to χ the system spends a lot of time in the minima's of in the oscillation (These are metastable states).

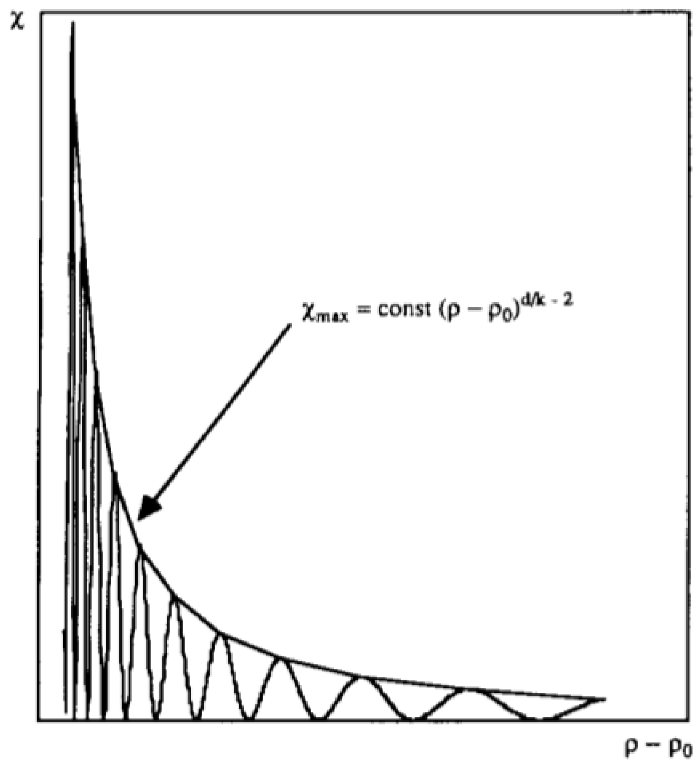

FIG. 2: Susceptibility for $n = 0, 1 < d/k < 2$. Figure taken from [4]

Further analysis of general limit cycles in RG by Hammer and Platter show that discrete symmetry invariance is required for limit cycles where $x \to (S_0)^n x$ with n an integer. [5]

V. EFIMOV PHYSICS

The Efimov effect was a prediction by V.N. Efimov in 1970 about an infinite set of bound states of three bosons. This bound state can occur even when the two body attraction is too weak to form a two body state. Additionally, the ratio of binding energies of two of these states is a universal number 515. The Efimov effect describes a discrete scaling symmetry in the three body system with a large two body scattering length a.[5]

Limit cycles in the Renormalization Group are used by Hammer and Platter as an alternative formulation of the Efimov effect. In the case of the Efimov effect the preferred scaling factor S_0 is 22.7. When the scattering length of two body interactions is taken to be infinity you find a three body bound state with energies differing by $(S_0)^2 = 515$. Also for limit cycles to be possible a discrete symmetry is observed in the Efimov effect of form $x \to (S_0)^n x$ with n an integer.[5]

Hammer found the three-body scattering threshold has a geometric spectrum as shown in Figure 3. This specturm is a result of the discrete symmetry that has scaling factor 22.7. Hammer showed these results by mapping the Efimov problem onto a Schroedinger like equation with a α/r^2 potential. An equation of this form can have a limit cycle if $\alpha < \alpha_*$. Additionally it has a preferred scaling factor of $S_0 = \exp(\pi/\sqrt{\alpha - \alpha_*})$.[7] In this way Hammer found that $S_0 = \exp(\pi/s_0)$ where $s_0 = 1.00624$ and thus $S_0 \approx 22.7$.

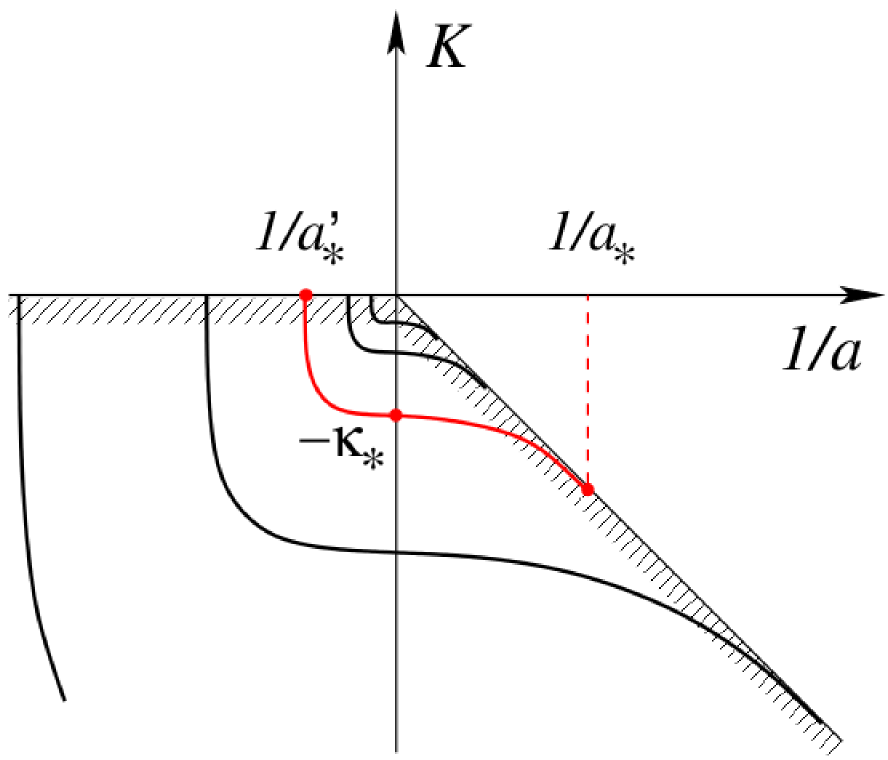

FIG. 3: Efimov plot. Plot of K_* the binding wavenumber of an Efimov state labeled by n versus inverse scattering length. Figure taken from [5]

The Efimov effect was originally a description of three nucleon systems. Now an analogous effect can be found for ultra cold atoms. Ultracold atoms are a great system to test the Efimov effect since it is possible to use Feshbach resonances to experimentally tune scattering lengths. The first experimental evidence of an Efimov state was in 2006 by Kraemer. He observed ultracold Cs atoms in their lowest hyperfine spin state. Kraemer observed a feature in the resonance of loss of atoms which could be attributed to an Efimov trimer crossing the three atom threshold. According to Hammer this can occur when the scattering length equals $(S_0)^n a$ with n an integer.[5]

The previous experiment only gives indirect evidence for the Efimov effect through observing atom loss. More recently experiments have directly observed the Efimov state through radio spectroscopy. Lompe found a three body spin state of ^6Li. Apparently the binding measurements of this state agree with theoretical predicitons.[5]

VI. RESULTS AND DISCUSSION

In this paper I have discussed some possible consequences for a limit cycle in RG flows. A limit cycle in an RG flow will correspond to a discrete symmetry invariance, also log periodic observables. Limit cycles can be applied to the Efimov effect to explain the geometric progression of scattering threshold and binding energies. Limit cycles are characterized by a preferred scaling factor S_0. In the Efimov effect the preferred scaling factor is 22.7.

Since limit cycles have been found in the RG of physical systems it is important to consider this possibility when performing an RG calculation. Other exotic behaviors are possible, including chaos.

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[1] K.G. Wilson, Phys. Rev. D, 3 (1971), p. 1818

[2] Goldenfeld, Nigel. "Lectures on Phase Transitions and the Renormalization Group" Westview Press, 1992.

[3] Strogatz, Steven. "Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemisty, and Engineering". Westview Press, 1994

[4] Boris A. Veytsman, Limit cycles in renormalization group flows: thermodynamics controls dances of space patterns, Physics Letters A, Volume 183, Issue 4, 13 December 1993, Pages 315-318, ISSN 0375-9601, 10.1016/0375-9601(93)90463-A. (http://www.sciencedirect.com/science/article/pii/037596019390463A)

[5] Hans-Werner Hammer, Lucas Platter Efimov physics from a renormalization group perspective Feb 2011. 22 pp. Phil.Trans.Roy.Soc.Lond. A369 (2011) 2679 e-Print: arXiv:1102.3789

[6] Glazek, Stanis. Limit cycles of effective theories, Phys. Rev. D, 75, Jan 2007 http://link.aps.org/doi/10.1103/PhysRevD.75.025005

[7] H.-W. Hammer, Brian G. Swingle, On the limit cycle for the 1/r2 potential in momentum space, Annals of Physics, Volume 321, Issue 2, February 2006, Pages 306-317, ISSN 0003-4916, 10.1016/j.aop.2005.04.017.(http://www.sciencedirect.com/science/article/pii/S0003491605000667)