Self-organized criticality in sandpile models

Arka Banerjee

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Abstract

The sandpile model, introduced in 1987, was the first model to exhibit self-organized critical behavior, that is, the system moved towards its critical point without the need to tune any adjustable external parameter. In this paper, we look at the why these models exhibit such non-intuitive behavior. We also look at some of the phenomenology near the critical point, such as scaling laws and critical exponents. Finally, we look at some experimental realizations of the sandpile model.

1 Introduction

In most systems which exhibit critical phenomena, there exist adjustable parameters which have to be finely tuned in order for the system to reach the critical point. For example, in the Ising model, one needs to adjust both the temperature and the external magnetic field to close to their critical values. In the language of the Renormalization Group, these are the relevant directions of the critical fixed point that we are interested in. If such a system is initially prepared such that the temperature and the external magnetic field are far from their critical values, the system will not show critical behavior even if other parameters are changed, or if the system is perturbed. Since most systems with phase transitions exhibit the above behavior, it would seem reasonable to guess that critical phenomena can be triggered only if all relevant parameters have been fine tuned.
However, in 1987, Bak, Tang, and Wiesenfeld introduced the *sandpile* model, which displayed spatial and temporal power laws and scale invariance, without controlling the external parameters. The evolution of the system was such that it spontaneously moved towards the critical point. Because of this, the critical behavior exhibited by this model was termed as *self-organized criticality*.
Self-organized criticality was gradually understood to be a feature of out-of-equilibrium systems with a slow driving force. These models of SOC provide a mechanism which can be used to explain the emergence of complexity in many natural phenomena. The behavior of such systems is unlikely to be governed by the fine-tuning of parameters, and the complexity must arise from the evolution of the system itself. In their original paper, Bak et al. claimed that the ubiquitous $1/f$ can be explained in terms of SOC. A wide range of other natural phenomena, such as naturally-occurring fractals, earthquakes, rainfall patterns, have all been investigated in terms of these models of self-organizing behavior spontaneous critical phenomena. In fact, the model has also been used to analyze systems which have no connection to physics, such as stock markets and sociology.
In this term paper, we will look at the sandpile model and its attendant critical behavior. In Section 2, we shall define the sandpile model and the rules of its evolution. In Section 3 we shall look at the emergence of scale invariance and power law behavior in the model. Section 4 looks at how SOC seen in sandpile models are related to a real phase transition with absorbing states. Section 5 looks at some experimental realizations of the model, as well as some natural phenomena to which the model has been applied. Finally, in Section 6, we summarize the main points and look at the current research on the field.

2 The sandpile model

2.1 Sandpile model in 1 dimension

Consider a one-dimensional array of sites of length L. At each site, we can place grains of sand, one on top of each other. With each site on the array, we associate a number $z_n = h_{n+1}$, which measures the difference in height between nearest neighbors. Adding a grain of sand at site n can then be represented as

$$\begin{aligned} z_n &\rightarrow z_n + 1 \\ z_{n-1} &\rightarrow z_{n-1} - 1 \end{aligned} \tag{1}$$

When z_n at a particular site crosses a critical value z_c, then one unit of sand topples down to the adjacent site

$$\begin{aligned} z_n &\rightarrow z_n - 2 \\ z_{n\pm1} &\rightarrow z_{n\pm1} + 1 \end{aligned} \tag{2}$$

The boundary conditions are such that grains of sand can only topple out of the system on the right, that is, at the Nth site. The other boundary is closed. If we start with all the sites empty and then add grains of sand at random, is there some configuration toward which the system will always tend?

It turns out that the final configuration is always the one where all the z_n

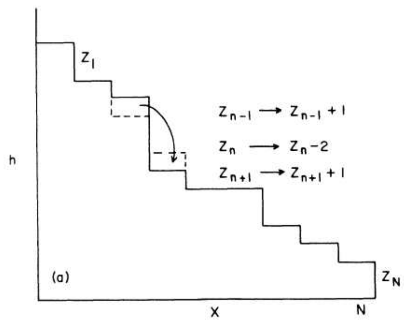

Figure 1: Sandpile model in 1 dimension. Taken from [2]

have their critical values, and the heights are in descending order from the closed boundary to the open boundary. We can see that if we add one more grain of sand to any of the sites, it will topple from one site to the other, until it rolls off the final site and out of the system. So this final state is stable with respect to perturbations and is called the minimally stable state. This gives us a rough idea of how a system can evolve under its own laws to a state where a perturbation propagates throughout the system (the grain of sand can be added anywhere and it will travel until it falls off the system). This is the sort of scale-free behavior one associates with critical phenomena.

2.2 Sandpile model in 2 dimensions

Most studies of the sandpile model was done in 2-dimensions, and we now need to set up the rules governing the dynamics in 2 dimensions. Instead of a 1D array, we now have an $N \times N$ grid, and we can write down the addition and toppling rules by modifying Eqs (1) and (2). The equivalent of Eq. (1), is now

$$\begin{aligned} z_{x,y} &\rightarrow z_{x,y} + 2 \\ z_{x-1,y} &\rightarrow z_{x-1,y} - 1 \\ z_{x,y-1} &\rightarrow z_{x,y-1} - 1 \end{aligned} \tag{3}$$

On the other hand, the toppling rule when $z_(x, y)$ exceeds the critical value is

$$\begin{aligned} z_{x,y} &\rightarrow z_{x,y} - 4 \\ z_{x\pm1,y} &\rightarrow z_{x\pm1,y} + 1 \\ z_{x,y\pm1} &\rightarrow z_{x,y\pm1} + 1 \end{aligned} \tag{4}$$

In the two-dimensional case, we work with free boundary conditions on all sides. Any grain crossing the boundary is lost. If we add a grain of sand to a site which is at the critical value of $z_(x, y)$, we trigger off an avalanche onto the adjoining sites. If these sites themselves are at the critical value, then the avalanche propagates, else it stops. A very important rule which one must obey while adding subsequent grains of sand, is to allow all possible avalanches to occur within the system before one adds another grain. In physical systems, this would correspond to the situation where the frequency of the driving force is small compared to the relaxation time-scale, that is, when the system is driven weakly away from equilibrium.
Unlike the 1 dimensional case, the configuration where all $z_{x,y} = z_c$, is not stable to perturbations, nor is it the final configuration starting from some random configuration. Instead, all that can be said about the system is that, given sufficient time, it will reach a steady state where on the average, the number of grains added to the grid is the same as that lost at the boundaries.

3 Scaling laws in the sandpile model

In this section, we look at the scaling behavior in the avalanches of the 2 dimensional sandpile model. This scaling behavior is similar to the one exhibited by systems close to a continuous phase transition, and arise from spatial and temporal scale invariance. It is important, however, to remember that in the sandpile model and its variants, there is no real phase transition.

As we saw in the last section, once the slope at a point on the grid exceeds

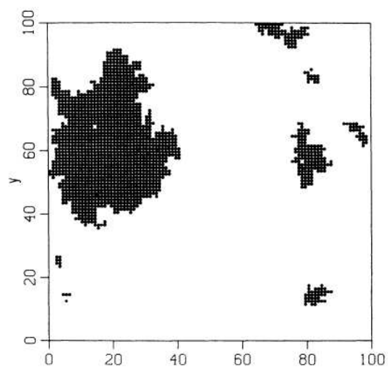

Figure 2: Avalanches in 2D sandpile model.The dark regions show the size of avalanches triggered by a single addition. Taken from [2]

the critical value z_c, it sets of on avalanche, with grains toppling onto the adjacent sites. These avalanches can be parameterized by three variables:

- the number of topplings s
- the area affected by the avalanche a
- the duration of the avalanche T

Though at first glance, it seems that the number of topplings, s and the area a both measure the number of affected sites, one must remember that a single site may topple more than once in a single avalanche, and hence the two are truly different variables. The duration of the avalanche is defined as the number of updates one must perform on the system before all the sites become stable after the addition of one grain of sand.

From numerical simulations of the 2D sandpile model, it was found that all of these variables have the following probability distribution

$$P(x) = x^{-\tau_x} G(x/x_c) \tag{5}$$

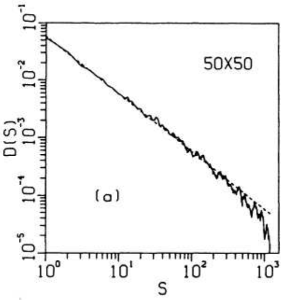

Figure 3: Scaling behavior in the number of topplings in the 2D sandpile model. Taken from [2]

Here, τ_x is the critical exponent associated with the variable x. $G(y)$ is a scaling function analogous to the scaling functions one writes down for conventional critical phenomena. x_c represents the cutoff for the system, which is naturally determined by the size of the system. We are interested in the asymptotics of the scaling function when its argument is small. In this regime,

$$G(y) \sim e^{-y} \tag{6}$$

As the system size (L) diverges, the cutoff x_c itself has a power law behavior, $x_c \sim L^{\beta_x}$. This is analogous to finite-size scaling in conventional systems, where the critical exponents are modified due to the system size being finite. β_x is also referred to as the fractal dimension. If one looks at the probability distribution for the time duration of the avalanches, the exponent β_T, which gives the dependence of the time cutoff on the spatial dimensions of the system, can be identified to be the same as the dynamical critical exponent z.
The fact that the various parameters of the sandpile model exhibit power law scaling implies that, like systems near a second-order critical point, we have scale invariance. Unlike systems in thermal equilibrium, the scale invariance in these models of SOC are temporal as well as spatial, as exhibited by the scaling of the time duration of the avalanches. This means that perturbations at a given point in space and time can propagate at all length and time scales, just as perturbations to systems near a second-order phase transition propagates at all length scales due to the divergent correlation length.
The exponents τ_x and β_x were used to define the universality classes of various sandpile models and their derivatives. Since these exponents had to be extracted from numerical data, they were difficult to pin down accurately, and so it was difficult to identify the universality classes of different models. In order to better distinguish between various universality classes, one has to perform a more involved treatment using the expectation values of the various parameters, while holding another parameter fixed. This is defined

in the following manner

$$E(X|Y=y) = \int dx\ x\ P(x, Y=y) \tag{7}$$

It was assumed that these expectation values also follow a power law near the steady state, $E(x|y) = y^{\gamma_{xy}}$. It can be shown that these exponents $\gamma_{x,y}$ must satisfy

$$\gamma_{xy} = \gamma_{yx}^{-1} \tag{8}$$

$$\gamma_{xy}\gamma_{yz} = \gamma_{xz} \tag{9}$$

Once these exponents have been found out, they can be used to relate the cutoffs in the distributions of different variables. This follows simply from dimensional analysis, by which one expects $x_c \sim y_c^{\gamma_{xy}}$. This means that

$$\gamma_{xy} = \frac{\beta_x}{\beta_y} \tag{10}$$

Since the cutoff for the distribution of area is expected to scale as L^2, one can then extract β_s and β_T. One can also write down the relationships between various scaling exponents like the scaling laws of regular critical phenomena. In 2D, the above analysis yields

$$\tau_a = 1 + 2(\tau_s - 1) \tag{11}$$

$$\tau_T = 1 + \frac{2(\tau_s - 1)}{z} \tag{12}$$

where z is the dynamic critical exponent.
In this section, we have seen how scaling laws are observed in the avalanche distributions of the sandpile model. These distributions are characterized by power law distributions and scaling functions, which point a scale invariance in the system. We also saw the relationships between the different scaling exponents, analogous to scaling laws.

4 The connection between SOC in sandpiles and a true phase transition

In this section, we shall look at connection between a variant of the sandpile model called the Manna sandpile model and a true second order phase transition in a model with absorbing states. The Manna sandpile model has different rules than the one we outlined in Section 2, but numerical analysis seems to show that both models lie in the same universality class. In this model, at every iteration, all sites with $z_{xy} \geq 2$ release two grains to randomly chosen nearest neighbors. A grain of sand is added once all the updates have taken place, and there are no more sites with $z_{xy} \geq 2$ left on the lattice. On the other hand, the model with a true phase transition that we will be looking at is called the activated random walker model, which we shall discuss below.

We consider a periodic lattice with L^d sites, where each site is occupied by z_{xy} random walkers. The total number of random walkers is taken to be N. Each random walker performs a random walk independent of the others. However, if a particular site is occupied by only one random waler, it is inactive until another walker falls onto the site and activates it. In this model, if $N > L^d$, then there will always be some activity on the grid. There will be at least one site where the occupancy $z_{xy} \geq 2$. On the other hand, if $N \leq L^d$, there is the possibility that the entire grid might become inactive. This is called an absorbing state. If the density of random walkers $\zeta = N/L^d$ is small, it is likely that for any initial configuration, the system will end up in an absorbing state. The question is, what happens as we increase the density, from 0 up to 1?

One can perform a mean-field calculation in the model by decoupling nearest neighbors and ignoring correlations. This mean field solution shows that there is a continuous phase transition in the system at $\zeta_c = 0.5$. Above this critical density, the probability of the absorbing state becomes so small that the walkers can be active indefinitely. Below the critical density, any random initial configuration will end up in an absorbing state. A more refined analysis gives the critical density to be $\zeta_c = 0.75$. One can also go ahead and calculate the critical exponents in the theory.

One can see that this model with the activated random walkers and the Manna model have similar local rules. However, one must remember that in the Manna model, one can add grains of sand by hand, whereas grains may be lost at the boundary. The random walk model, with periodic boundary conditions, does not include such a possibility. In the language of the ARW model, the Manna model adds one grain of sand to a random site every time an absorbing state has been reached. Comparing the two models, one can see that the in the steady state of the Manna model, the density of the grains tends towards the critical density of the ARW model. The addition and loss of particles is then balanced, and the steady state, as the name suggests, is stable to perturbations.

One can therefore, go from a model with a true second-order phase transition (the ARW model) to the Manna model in the following manner. The relevant variable, which has to be tuned in the ARW model is the particle density ζ. Suppose one starts with the value of ζ just below the critical density ζ_c. Given sufficient time, the system will become inactive. Suppose one adds a walker to the system. The density now becomes $\zeta \to \zeta + 1/L^d$. Suppose that the addition of one walker does not make the density reach its critical value. Then once more, after sufficient time, the system will become inactive. On the other hand, if the addition of the walker pushed the density of the critical density, the system would become active, and at some point, some of the walkers would reach the edge of the grid. We now allow the walkers to be lost at the edges. The rate of loss of walkers is given by $d\zeta/dt \propto -L^{-1}\rho_b$, where ρ_b is the number of sites at the edge with $z_{xy} \geq 2$.

If the value of L is not large enough, the addition of one walker changes the density by a finite amount. Above the critical density, the loss rate will also be large for the same reason. So, if one starts off just below the critical density, the ARW system will keep jumping from being overcritical to being undercritical without sitting at the critical point, where the Manna model sits. To ensure that one does indeed reach the Manna model, one has to take the $L \to \infty$ limit, which is basically the thermodynamic limit. In the thermodynamic limit, the addition of a walker changes the density infinitesimally, and the loss rate is also infinitesimal. The system can then reach the critical point, and sit there as the gain and loss rates both go to 0. At this point, the ARW model becomes equivalent to the Manna model in its steady state.

It is interesting to note here that in the Manna and other sandpile models, one has to wait for all avalanches to occur before adding another grain. This means that the model needs external supervision, thereby reducing its credibility as a candidate for explaining true self-organized behavior in nature. This point can be circumvented by looking again the how the ARW model at its critical point becomes equivalent to the steady state of the Manna model. It was postulated that the rate of addition is actually continuous, rather than occurring at discrete time-steps. Self-organized critical behavior takes place when the rate of addition $h \to 0^+$. If follows that the loss rate, ϵ should also go to 0. However, to ensure that all avalanches take place within the system before the addition of a new particle, one needs to ensure that $h/\epsilon \to 0$.

Even though the above process obviates the need for an external agent to add particles at the right time, it comes with the price of introducing tunable parameters in the system, which are the addition and loss rates. Both of these parameters have to be tuned to 0 in the manner specified above for the system to exhibit SOC. It seems, therefore, that sandpile models may not be as general models of SOC as they were originally thought to be. However, certain natural systems do have slow rates of addition and loss, and for these systems, the sandpile model still serves as a good analysis tool of SOC.

5 Experimental realizations of the sandpile model

Till now, we have looked at the sandpile model as a theoretical construct, and most of the results that have been outlined earlier have been derived from computer simulations. In this section, we will take a look at some of the experimental realizations of the model.

The earliest experiments, in 1989, used real grains of sand to perform the experiment. Though great care was taken to eliminate external sources of error, such as the exact position where the grain had been added, and the exact rate of addition of sand, the results did not match with the ones which had been predicted from the original BTW paper. In particular, one did not observe power law distributions in the size of the avalanches. It was then understood that the experiment had failed to take into account the properties of real grains of sand. The sandpile model treats the sand grains as idealized particles which hop from site to site without any forces acting on them. On the other hand, in a real system, factors such as friction and inertia play a role. In particular, grains of sand exhibit a sort of hysteresis effect. This means that even when the slope exceeds the predicted critical slope, the particles do not move due to static friction. Once the slope has been increased further beyond the critical value, the particles start moving. This meant that the system behaved as if it is close to a first order or discontinuous transition rather than a continuous transition.

To overcome these problems, experiments were performed with various other materials. It was realized that by changing the shape of the grains, and making them more elongated, rather than round, one achieved better results. Finally, in 1996, a group in Oslo performed the experiment with grains of rice. They used different types of rice, so that they could vary the aspect ratio (ratio of the thickness to the length of a grain of rice). They also used rice grains with different degrees of surface smoothness to have a handle on the role of friction in the experiments.

The experiment was set up with two vertical plates placed parallel to each other. The distance between the plates was adjusted so that it was of the order of the width of one rice grain. So essentially the experiments studied the sandpile model on 1 dimension. However, it was found that varying this width did not have a strong effect on the results. One of the edges of the

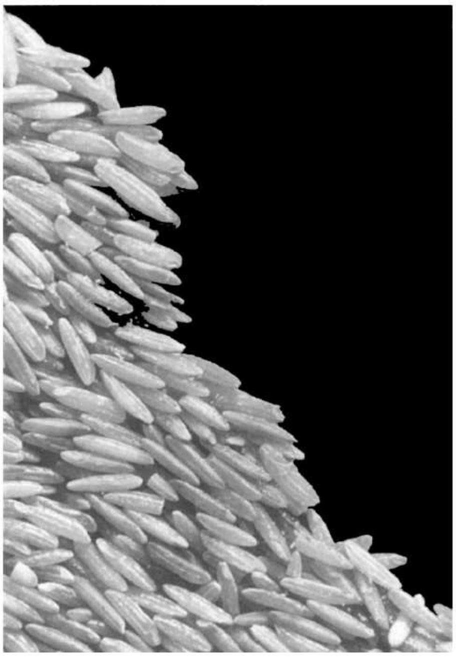

Figure 4: Profile of the slope for rice grains with large aspect ratio. Taken from [8]

setup was kept open, and grains were added at the other edge. The experiment used a CCD camera to look at the profile after fixed time intervals (15s), and avalanches were studied by looking at successive temporal images and comparing the local slopes.
It was found that if one used rice which had a smaller aspect ratio, that is, roundish grains, the experimental results did not match with the predictions of the sandpile model. On the other hand, grains with large aspect ratios did indeed show the expected scaling behavior. It was also observed that for grains with large aspect ratio, the smoothness of the surface did not make much of a difference to the avalanche distributions. This is because for round grains, the particles tend to roll down the slope whereas for flatter grains they slide down the slopes. The rolling grains acquire a larger kinetic energy and this energy dominates over the local interactions which are responsible for the behavior of the sandpile model. The longer sliding grains have lower kinetic energy and are hence more affected by local interactions.
The results from the experiment showed that by using elongated grains, one could get a realistic approximation of the predictions of the sandpile. Later, similar experiments were performed to study a two-dimensional realization of

the sandpile model. The SOC behavior of sandpile models was also studied in other more exotic experiments, such as superconductors in magnetic fields.

6 Summary and Conclusions

In this term paper, we looked at a model for self-organized criticality, that is, systems which show features such as scale invariance and power law scaling without the tuning of external parameters. In Section 2, we introduced the model, called the sandpile model, and the rules for its dynamic evolution. We saw how avalanches can arise in the system when the slope at any point exceeds a critical value. In Section 3, we saw the power law scaling in various observables related to these avalanches, such as their size, temporal length, and the number of sites affected by them. We also saw the analogues of the scaling laws, which relate the different scaling exponents.

In Section 4, we tried to understand the connection between self-organized criticality and a real second order phase transition. We studied the Activated Random Walker model to see how it becomes equivalent tho the steady state of one class of sandpile models, known as the Manna sandpile model, at the critical point. It was seen that self-organized critical behavior can be studied by looking at absorbing state phase transitions. The mapping also gave us a new perspective on SOC by showing that there does indeed arise adjustable parameters in the model, if one has to do away with external supervision which is implicit in the laws of the dynamic evolution of the model. Specifically, this had to do with the fact that one has to wait for all possible avalanches to occur before one adds a new particle to the system.

In Section 5, we looked at how an experimental realization of the sandpile was achieved. We saw how the initial attempts with real grains of sand failed due to effects such as inertia and friction. We then saw how an experiment with rice grains, which reduced the effects of inertia on grains rolling down the slope, managed to validate the predictions of the sandpile model.

Though questions have been raised about the validity of using the sandpile model to study the most general manifestations of self-organized criticality, it remains a widely used tool in various areas of physics as well as other subjects. In fact, many other similar models have been proposed over the years to explain the emergence of scale invariance in many real-world systems, and is an open research subject which has received a lot of attention, especially in the recent past.

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