Physics 563 Term Paper

Robustness of interacting networks

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Abstract: In the real world the systems are composed of networks which are coupled together. The analytic works studying the robustness of a system of interdependent and interconnected networks under failures or attacks are reviewed. Through the generating formalism for percolation process, in specific analytic models, the final fraction of functional nodes in the networks is found to have unusual transition between first and second order phase transitions as a function of the number of interdependent networks, the initial fraction of the remaining nodes and the dependency of couplings between interacting networks.

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1 Introduction

Complex networks are usually used to describe the interacting behaviors of multi-agent systems in social science, bology and technological communication. Studies on the statistical properties of complex networks with different structures, or topologies, and distribution of functionality of elements are widely applied in these fields, such as the Internet, the spread of epidemic disease, the traffic networks, protein-protein interactions and polymer networks [3, 7].

The first model of networks was proposed by Erdös and Rènyi (ER) [13]as the standard random graph theory. In their model each pair of elements is randomly connected with the same probability, and thus the distribution of the edges, or links to the neighbors, of nodes becomes Poissonian. While in real world, connectivity between each element could be various. In fact, studies showed that the standard random graph theory does not accurately predicted the real networks, but it is still a simple model that captures the properties of the performance of the real networks [2]. Particularly, random graph models with certain distributions can be exactly solvable and are found properly describe the behaviors of networks [2, 8]. It is found that many real networks, such as the Internet and social networks, can be approximated well if the distribution of degrees of nodes follows a power law, which are so-called scale-free networks [3, 7].

1.1 Stability of networks and the emergence of giant component

When networks form, one of the essential questions is their stability. The stability, or the robustness, of the network is described by the permeability of the network. When certain nodes of the network face failures, the transmission of failures changes the symmetry of the functionality of the network, and a network can be separated into pieces of clusters consist of several connected nodes. After steps of transmission, or the cascading of failures, the largest cluster emerges. Conventionally, the transitivity of one node is assumed to be blocked out (isolated) if it is not connected to the largest cluster. The largest cluster of the network is usually called the giant component, which is one of the characteristic statistical properties of networks.

2 Percolation model to approach the resilience of interacting networks

The random graph model leads to a critical probability where the giant component emerges [7, 8]. Above the critical probability, the giant component exists and the function of the network works, while below the critical transitivity the network turns into pieces of small, isolated clusters and the network shuts down. This is analogous to behaviors across the critical probability, or the percolation threshold, for the percolation transition, and the giant component in the random graph model is similar as the percolating cluster over the system. Therefore, the percolation language is usually applied to study the robustness of networks, where the critical transitivity can be considered as the fraction of the normally functional elements, p, under attacks. The critical value of p is regarded as the measure of robustness, and the size of the giant cluster is the order parameter. For less p_c, the network is easier to be prevalent and thus is less fragile.

The standard percolation process undergoes a second order phase transition[7, 2], where the giant cluster varies continuously, and sharply if the size of the system is infinite, with the transitivity p. This indicates that the functional fragment of elements can be extremely small, comparing to the network size, as approaching the threshold.

2.1 Interactions within and between networks

In the percolation model, the elements are connected together and the transitivity of a network is decided by those connective links. Nevertheless, in real world, elements in the networks can be connected and also depend on the functionality of each other which may not be connected together, i.e. not necessary to be local effects [1, 3, 4, 5, 6, 9, 11, 12, 10]. The elements functionally depending on the each other form the dependent groups (clusters) in a network. When an element fails, its connected links to the other elements are destroyed. Also, the other elements in the same dependence group of that node will fail but are still connected to some other nodes. For example, for a market network composed of different stores, if a store shuts down the other stores having business with it are affected, but the other stores of the same company as that store will also seriously influenced since they are highly dependent. When it comes to different networks coupled together, nodes between networks can also be bonded connectively and dependently. Consider an airport network and a railway network, where airports are connected to other railway stations. If one airport fails due to some accident or attack, the connections of the railway stations connected to that airport are also blocked, but if the airport traffic is broken, the railway traffic is also diabled.

2.2 Generating function formalism for the calculation of the giant component

A systematic way to discuss both connective and dependent properties is to consider percolation process with dependence links to other elements in the system [1, 3, 4, 5, 6, 9, 11, 12, 10, 8]. The percolation process can be analytically described by the generating function formalism. First we consider a single network. The formalism can also applied to a system composed of many networks. By introducing the generating function of the degree distribution,

$$G(x) = \sum_{k=0}^{\infty} P(k)x^k$$

where $P(k)$ is the probability of one node with k outgoing links, i.e. with a degree k, the average degree of the nodes in the networks, $\langle k \rangle$, can be described as $G'(1)$. Similarly, after reaching a node with a degree k, there are $k-1$ outgoing links to next other nodes, and the probability of another node connected to the first node is $kP(k)/\langle k \rangle$. Then the new generating function describing this connectivity process, or branching process, can be expressed as

$$G_1(x) = \sum_{k=0}^{\infty} P(k)kx^{k-1}/\langle k \rangle = G'(x)/G'(1)$$

To study the cascading failure process, now suppose the probability of any link which is not connected to the giant component is f. If a fraction $1 - p$ of nodes in the network is removed or fails, for an arbitrary node, the probability to connect to the giant component is $p(1-f)$ and the probability to disconnect to the giant component is thus $1 - p(1 - f)$. This can also be described by the branching generating function, $G_1(f)$, which is just the probability that a node with $k-1$ out edges that is not connected to the giant component. Therefore, the self-consistent equation $f = G_1(f)$ can be solved. Then the probability that a node is connected to the giant component is $g = 1 - G(1 - p(1 - f))$, and the giant component after this connectivity process is $P = pg$.

The dependence process within a network can be characterized by the dependency groups with a probability distribution $q(s)$ of size s, where $s \geq 1$. After a fraction $1 - p$ of nodes removed, the number of functional nodes is $\sum q(s)Np^s$, and the fraction out of the unmoved nodes becomes $\sum q(s)Np^{s-1}$, where N is the size of the network. Combined the analytic conditions of connectivity and dependency process, the steady state of the network can be solved with numeric [10].

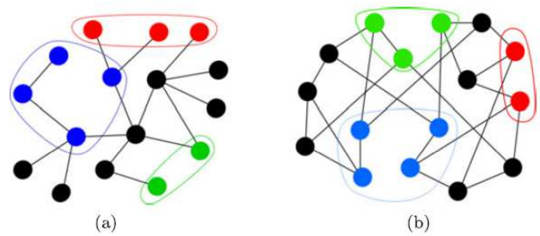

Figure 1: Different dependency groups in networks [10]. The nodes of the same depenency group have the same color and circled together. The nodes in the same dependency group crucially depend on each other but are not necessarily connected together.

Basha *et al*. [11] studied a single network contains Poisson distributed connectivity links and dependency links trough the analytic and numerical approach above. They found if there are only connectivity links between nodes, a second order phase transition of the size of the giant component occurs when increasing the fraction of the initial fail nodes, which is similar as the observed connectivity process in the traditional percolation model. However, when dependence links exist, they will induce a first order phase transition instead, indicating a quanlitative, sudden change of states. The discontinuous evolution of the giant component means that due to dependency links the critical failure fraction which leads to the breakdown of a network is finite. In fact, the first order transition happens at a lower fraction of initial failures, which means that the dependency interactions make a network more vulnerable. Moreover, when the sizes of dependency groups increase, the critical initial fraction of failures decreases and the network

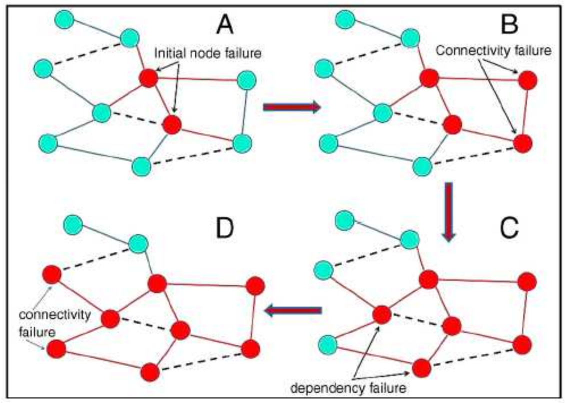

Figure 2: The cascading process of failures in a single network with both connectivity and dependency links [12]. The solid bonds are connectivity links, while the dependency links are characterized by dashed lines. The red dots are fail nodes. During the connectivity process, or the percolation process, nodes connected to the initial failures but not connected to the giant component (the largest cluster) are damaged. Similarly, in the dependency process, the nodes crucially depend on the fail nodes also fail. Therefore, the size of the giant component evolves during the cascading process of failures.

becomes less robust. This could be understood by considering different transitivity rules in percolation processes, such as the explosive percolations.

3 The analytically solvable examples of the network of networks

In general, the degree distribution, or the structure, and the interactions of the networks are various, and thus after attacks the evolutions of the symmetry of distinct networks can change and may lead to different phase diagrams.

However, the networks in real like are coupled together. Recently, simple examples for a system composed of n interacting networks with interdependent couplings and intraconnected couplings were studied. Applying the similar percolation theory approach, Gao *et al*. [1, 3] studied a the system consist of n ER networks with intraconnected and interdependent links. Particularly, if there are two networks A and B, the one-to-one interdependency can be described as a fraction q_A of nodes in network A and a fraction q_B of nodes in B. Next, let $g_A(\psi_i)$ and $g_B(\phi_i)$ be the fractions of A nodes and B nodes that are in the giant components after removing fractions $1 - \psi_i$ and $1 - \phi_i$ of nodes in each network per step i. Therefore, by applying the generating formalism above, the evolution of the giant components P_i^A and P_i^B can be derived during the cascading process. For example,

$$\psi_1 = p, \ \ \phi_1, \ \ P_1^A = \psi_1 g_A(\psi_1)$$
$$\phi_2 = 1 - q_B[1 - p g_A(\psi_1)] \ \ ...$$

By applying the distribution of node degrees, which was Possionian in this study, the final size of the giant components can be get from the stable condition as $n \to \infty$. The results showed that for a the system consist of n ER networks, the interdependent couplings between networks can also lead to a first order phase transition of the giant component as the intradependent couplings in a single network. The discontinuity of the giant component at a finite failure fraction indicates the interdependency also decrease the robustness of the networks. The intra- and interdependency could be analogous to the role of interactions between molecules in the ideal gas. The transition between first and second order phase transition is similar as the case in the interacting ideal gas [5]. Interesting results of transitions between first order and second order transition were also found for full and partial interdependency in some simple topologies, as described in Fig. (3)[1, 3].

Furthermore, based on the similar generating function formalism, Leicht *et al*. [9] studied two Poisson distributed intraconnected network with interconnected links. They found the fraction of functional nodes grows as the interconnectivity increases.

The above studies showed that interdependency and interconnectivity of two interacting networks compete with each other. To understand the competing effect, Hu *et al*. [6] applied the similar scheme to study two intra-connected networks but with both interdependent and interconnected links. They found transitions of first order between second order phase transitions of the giant components of each network, and interestingly, also a hybrid of the two kinds of phase transition in Fig. (5) [6].

4 Summary

The stability of the networks can be described by the dynamics of the cascading process of failures and measured by the size of the giant component. The cascading process is analogous to the percolation process. In general, there could be connectivity and dependency couplings between nodes in networks. How the evolution of the giant component is affected by the interactions within and between networks can be analytically and numerically solved by the generating function formalism in the percolation point of view. In particular cases, intradependency and interdependency were found to reduce the robustness of the system, while interconnectivity makes the system more stable. Also, the competing effects between dependency and connectivity couplings leads to unusual transition between continuous and discontinuous phase transitions and even a hybrid of them, which indicates the dependency and connectivity can both cause qualitative change of the symmetry of the networks. Since the recent studies most focused on Possionian random graph models, it is interesting to ask what will happen when this scheme is applied to systems of interacting networks with the arbitrary distribution of node degrees, especially power law distribution which is common in real world. Moreover, it is also possible to study the competing behaviors between intra- and interdependent interactions in a system of coupled networks.

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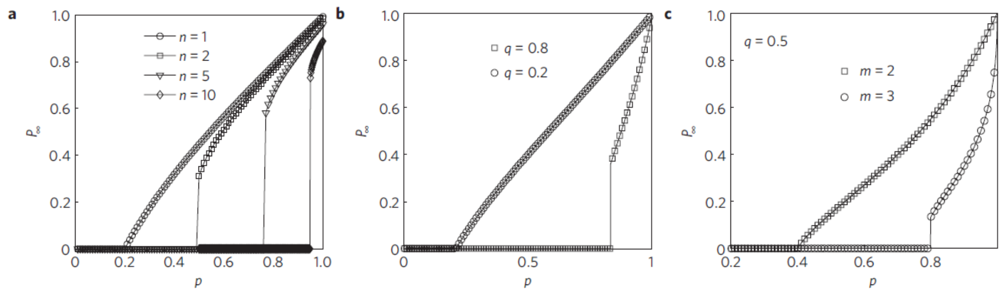

Figure 3: The fraction of giant components of n interdependent networks [3]. (a) The giant component of a system of networks with tree-like topology undergoes second order phase transition when n=1 but first order phase transition when n is larger than 1. (b) The example for a system of networks with loop-like topology. There is a transition between fist and second order phase transition depending on the interdependency q. (c) The example for a system of random regular ER networks. A transition between fist and second order phase transition also occurs as a function of m, the number of interdependent networks.

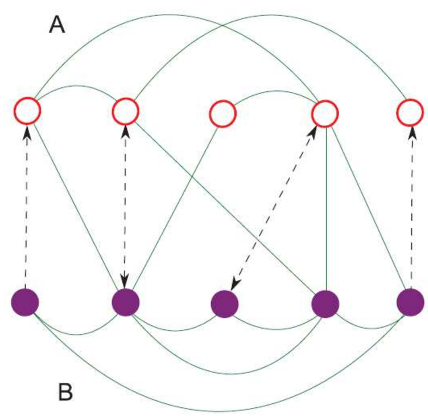

Figure 4: A network composed of two networks [6]. Solid lines are connected bonds and dashed lines are dependent bonds. Each network has intraconnected links, interconnected links and interdependent links.

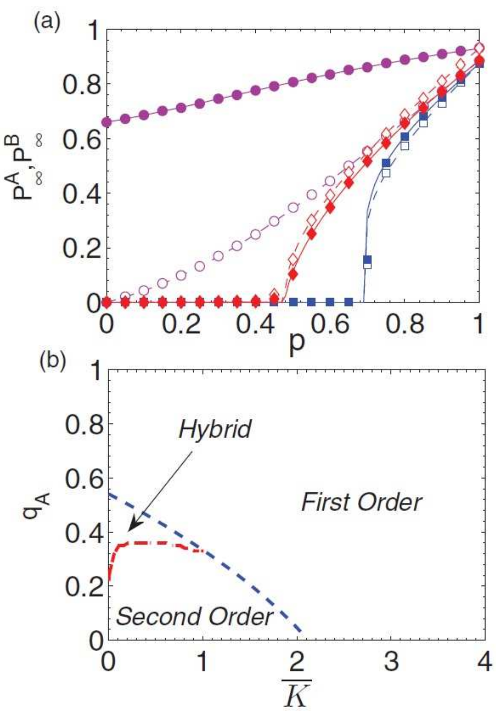

Figure 5: (a) The giant components of two networks containing interdependent and inconnected interactions [6] for fixed average intraconnected degree $k = 2$ and interconnected degree $K = 1$ and different interdepency q. There are three kinds of phase transions: no phase transition, second order and first order phase transition. (b) A hybrid of first and second order phase transition happens for $q_B = 1$ and average $k = 3$.