Turbulence as a Non-equilibrium Phase Transition in the Directed Percolation Universality Class

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Abstract

Fluid turbulence remains one of the most enduring challenges of classical mechanics, mostly because the underlying equations of motion of fluids are so difficult to solve in turbulent parameter regimes. Here, I review directed percolation, a simple cellular automaton whose nonequilibrium universality class is wide and applicable as a simple model of turbulent flow. The critical exponents of this universality class are calculated first in mean field theory and then in an ϵ-expansion.

1 Introduction and Motivation

The flow of an incompressible fluid is characterized by the Navier–Stokes equations, which can be derived simply from continuum mechanics and basic conservation principles such as momentum and particle number. The assumption that the Navier–Stokes equations are sufficiently complete to produce the complex phenomenon of turbulence has not gone unquestioned [1]. Nonetheless, the majority of work done in understanding the onset of turbulence has been founded upon the continuum theory of the Navier–Stokes equations.

The methods of studying turbulence became more statistical-mechanically oriented with Kolmogorov, Reynolds, von Karman, and more [2]. Turbulent flow took on a stochastic rather than deterministic character. Kolmogorov's well-known result for the power-law scaling of energy dissipation at many length scales describes isotropic turbulence after it has become well-established in the fluid. But to talk about the transition from laminarity to turbulence is entirely different. This is the interest of this review.

Restricting our picture of turbulent flow to Poiseuille flow in a pipe, the onset of turbulence is defined the by dimensionless Reynolds number $Re = vL/\nu$, where v and L are characteristic velocity and length scales of the flow and ν is the kinematic viscosity. Laminar flow is steady and reversible, but beyond a critical Re, turbulence appears spontaneously and relaxes back to laminar flow. As the Reynolds number becomes larger, turblence can become fully formed, and laminar flow is no longer a stable state. This is reminiscent of a phase transition with a critical point whose tuning parameter is given by Re. This formulation of the transition from laminar to turbulent flow as a *nonequilibrium* phase transition has lent itself to many kinds of statistical analyses [3, 4, 5].

For instance, Zubarev, Morozov, and Troshkin [3] take the standard approach of expressing the flow velocity field $\mathbf{U}$ as the sum of its ensemble average $\bar{\mathbf{U}}$ and a fluctuating random field $\mathbf{u}$ with mean zero. Making this substitution into the Navier–Stokes equations, an equation of motion for the average velocity is produced. The stochastic field manifests as an extra term in the stress tensor, $\rho\langle u_i u_j \rangle$. Upon further expansion in the field u_i, equations involving higher moments are generated, such that the nth moment depends on the $(n+1)$th moment: the "celebrated problem of closure [1]." At this point, the simplest thing one can do is incorporate phenomenological results from experiments or appeal to symmetries in specific boundary situations; Zubarev et al. tackle this problem in the context of Poiseuille flow in a stationary pipe.

A different formulation of the problem that has shown considerable success are lattice models of fluids. Pomeau [6] and Willis and Kerswell [5] explores the phenomenological behavior of turbulence in a pipe. For low Re, the flow is laminar and described by a parabolic velocity profile for an axially symmetric pipe. As Re increases, turbulent "puffs" form, but disappear after a finite lifetime. For large Re, the turbulent phase dominates, and a small perturbation produces a stable "slug" of turbulent flow, superceding the now unstable laminar phase [5].

Pomeau's idea is to characterize this process of turbulent spreading and relaminarization as an infection process not unlike the spread of disease in the model of the *contact process* (CP) [7]. In CP, a d-dimensional lattice has sites that are either "healthy" or "infected," and in the continuous time formulation, each healthy site that shares a nearest-neighbor bond with an infected site has a finite transition rate p to become infected. Infected sites transition back to being healthy with rate λ. The time-discretized formulation of the CP process is essentially the model of *directed percolation*, which Pomeau suggests can capture underlying dynamics of the Poiseuille laminar-turbulent transition.

Directed percolation (DP) forms a *nonequilibrium universality class*; it has a homogeneous *absorbing phase*, when all the individual cells are uninfected. The state is called absorbing because it is stationary in time; only unaffected cells next to infected cells have a probability of becoming infected [8]. Here, various methods to calculate the critical exponents of the DP universality class are explored.

2 Directed Percolation: The Model

To begin, first we consider *isotropic percolation*, which is an equilibrium model with no time dynamics. Consider an $L \times L$ lattice. Each cell that sits on a lattice site can be "occupied" or "unoccupied." The probability that a site is occupied is p. A cluster is formed when occupied cells are connected by nearest neighbor bonds. The question of percolation is what value of p guarantees that there is a system-spanning cluster? This critical probability p_c ensures the average cluster size becomes the system size.

In *directed percolation*, the process of placing occupation numbers on lattice sites is not static, and there is a dimensional anisotropy. A helpful analogy is to consider water trickling down a dry river bed. Here, the percolation is two-dimensional, as well. The percolation is directed downstream, but there is still lateral motion of the water as it moves between stones. The dimension along which the percolation is

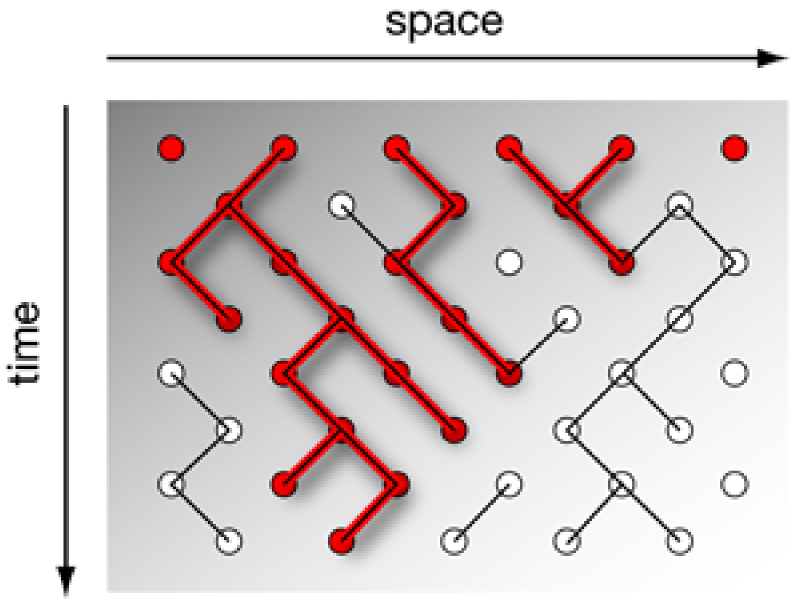

Figure 1: *Directed percolation:* a 2d array contains some occupied bonds (red) and unoccupied bonds (black). The bonds propagate downwards (forward in time) activating sites along the way. This is an example of *directed bond percolation* rather than site percolation. Image due to Hinrichsen [10].

directed is noted as $d_{\|}$, and the direction perpendicular to this dimension is $d_{\perp}$. It is useful to think of $d_{\|}$ as the temporal dimension and $d_{\perp}$ as one spatial dimension. Thus, this problem is $1+1$ dimensional [9].

The dynamical rules of DP are simple. If a site $r(x, t-1)$ is infected (denoted as a 1), then with probability p set $r(x+1, t)$ and $r(x-1, t)$ (independently) to 1 [11]. These are represented graphically in Figure 2.

In isotropic percolation, there is a correlation length between sites ξ which grows as a power law as the critical probability p_c is approached: $\xi \propto |p - p_c|^{\nu}$, where the exponent ν is universal. In (anisotropic) directed percolation, there are *two* correlation lengths to define, and each has its own universal scaling exponent. So there is $\xi_{\perp} = \xi \propto |p - p_c|^{\nu_{\perp}}$ and the correlation time $\xi_{\|} = \tau \propto |p - p_c|^{\nu_{\|}}$ [8].

The phase transition in directed percolation is similar to that in isotropic percolation; there is a spreading process that forms a cluster that spans the entire system, or, if the system is infinite, continues to grow without bound. To reconnect to the idea of turbulence, this would be a phase of steady "well-developed" turbulence [1].

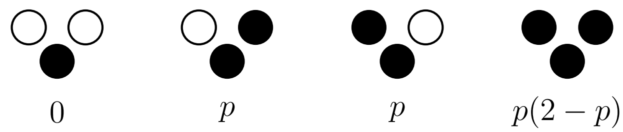

Figure 2: *Directed percolation:* the vertical dimension is time (increasing downwards) and the horizontal is space. Open circles are unaffected cells, and closed circles are affected. The probability of each iteration is shown beneath the diagram [9].

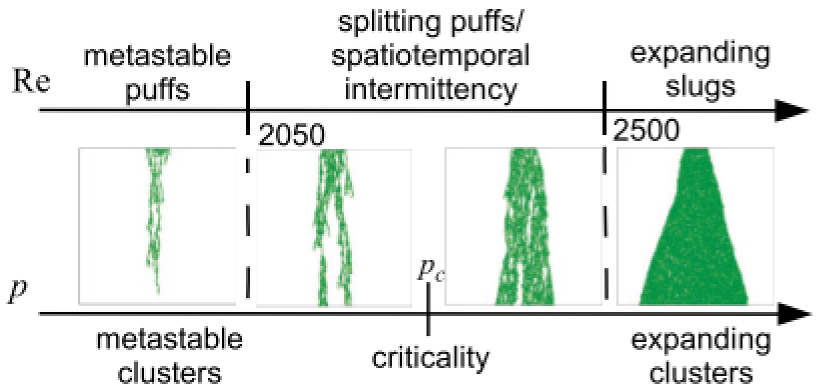

Figure 3: *Dependence on p:* shown is a typical simulation of directed percolation below, at, and above the critical probability p_c. Image due to Sipos [11].

The onset of this phase of perpetual infection growth, as p increases, is preceded by clusters that terminate after a finite time. These are the "puffs" that characterize transient turbulence.

Presented now are basic overviews of the techniques to compute the critical exponents and critical probability p_c. Then comparisons will be made to experiment.

3 First Step: Mean Field Theory

The simple mean field strategy is taken from [12], and it is reminiscent of the isotropic percolation p_c calculation in [13]. The probability that the site $r(x, t+1)$ is laminar (denoted as binary 0) is contingent on the probabilities that the sites $r(x+1, t)$ and

$r(x-1,t)$ are turbulent (denoted as binary 1) and did not infect it.

$$P(r(x,t+1)=0) = (1-pr(x+1,t))\,(1-pr(x-1,t))\,. \tag{1}$$

If both $r(x+1,t)$ and $r(x-1,t)$ are 0, then they have no probability to transmit to $r(x,t+1)$. Now we take an ensemble average over many configurations of the system, and define the probability that $r(x,t+1)=1$ as q.

$$1-q = 1-p\langle r(x+1,t)\rangle - p\langle r(x-1,t)\rangle + p^2\langle r(x+1,t)r(x-1,t)\rangle. \tag{2}$$

We interpret $\langle r(x_i,t_j)\rangle$ as the average probability that cell $r(x_i,t_j)$ is turbulent, i.e. a "turbulence density." Furthermore, we make the mean field theory assumption that each site is statistically independent (i.e. $\langle r_i r_{i+j}\rangle = \langle r_i\rangle\langle r_{i+j}\rangle$) and have the same probability to be turbulent, so $\langle r(x+1,t)\rangle = \langle r(x-1,t)\rangle = q$. Placing this in Equation 2,

$$1-q = 1-2pq+p^2q^2, \tag{3}$$

$$q=0, \quad q=\frac{2p-1}{p^2} = 2(p-p_c)-4(p-p_c)^2+\mathcal{O}(p-p_c)^3. \tag{4}$$

Here, the critical probability $p_c = 1/2$, where the two allowed solutions coincide, and below this critical probability, the average turbulence density q is 0, as the second solution is not physical. The transition is continuous, because at the critical point, the two solutions coexist at the same value, 0. Above p_c, there is a nonzero turbulence density solution, but $q=0$ is still a solution. That $q=0$ is still a solution defines it as the "absorbing state;" if the system begins with no turbulence, then the steady-state density of turbulence will always be 0. Above p_c this solution becomes unstable, and a turbulent "slug" propagates to fill the system.

4 Field Theory Renormalization Group

The DP model, to be made amenable to field theoretical calculations, must be coarse-grained and made continuous. Equation 1 expresses the form of the time evolution of the system. Letting the average turbulence occupation number of a site be equal to the coarse grained density q (as in Equation 2), but not imposing equilibrium $q(t+1)=q(t)$, we find the equation of motion for the field q:

$$\frac{dq}{dt} = 2pq - p^2q^2. \tag{5}$$

To account for spatial inhomogeneities, the diffusion term $D\nabla^2 q$ is included. Higher order derivatives have been shown to be irrelevant [10]. Finally, a Langevin noise term η is included such that $\langle \eta(x,t)\eta(x',t')\rangle = \Gamma q(x,t)\delta^2(x-x',t-t')$.

$$\frac{dq}{dt} = 2pq - p^2q^2 + D\nabla^2 q + \eta. \tag{6}$$

The form of the noise is such that it "respects the absorbing state" [9], i.e. if $q=0$ everywhere, then $\eta = 0$ for all x and t and the homogeneous state is stationary. The covariance of the noise is linear in the field q because the fluctuating generation and annihilation of occupation sites should be a Poisson process, whose variance is equal to its mean [9].

The noise term η can be replaced by an effective response field $\bar{q}$, the derivation of which is due to Janssen [14] and Ódor [8], and the effective action that describes the field q is

$$S = \int dxdt\, \bar{q}\left(\partial_t - 2p - D\nabla^2\right) q + p\sqrt{\frac{\Gamma}{2}} \int dxdt\, \bar{q}q^2 - \bar{q}^2 q. \tag{7}$$

The first integral describes the free action, and the second integral describes the action associated with interactions [15]. By writing the free part of the action in reciprocal space (in time and space), we can read of the two-point correlation function (referred to synonymously as the bare propagator in McComb [13]) and analogous to the calculation of the two-point correlation function in the Gaussian approximation [16]:

$$\left(G^0\right)^{-1} = Dk^2 - 2p - i\omega. \tag{8}$$

The renormalization of the bare propagator will come from the interaction terms. Following Adamek [15] and Hinrichsen [10], they make the "one-loop approximation" to the renormalization of the bare propagator. I won't reproduce their calculation, but I will generally describe the procedure and present the exponent results. They follow Wilson's standard procedure of dimension scaling and integrating out the short-wavelength degrees of freedom. First, let Λ be slightly less than one, and it will serve as the coarse-graining scaling factor.

$$x \to \Lambda^1 x \quad t \to \Lambda^z t \quad q \to \Lambda^\chi q \quad \bar{q} \to \Lambda^\chi \bar{q}. \tag{9}$$

Here, z is the time scaling exponent and is equal to $\nu_\parallel/\nu_\perp$ and χ is equal to $-\beta/\nu_\perp$ [10]. Under the transformations in Equation 9, the couplings and operators in the new rescaled action are

$$\partial_t \to \Lambda^{2\chi+z}\partial_t, \quad p \to \Lambda^{2\chi+d+z} p, \quad D\nabla^2 \to \Lambda^{2\chi-2+d+z} D\nabla^2. \tag{10}$$

The one-loop expansion and integration over the short-wavelength degrees of freedom in the shell $\Omega < |k| < \Omega/\Lambda$, where Ω is the ultraviolet cutoff from the DP lattice, renormalize the bare propagator:

$$\left(G^0\right)^{-1} \to \left(G^0\right)^{-1} - \frac{\Gamma^2}{2} \int d^dk' d\omega' \, G_0(k/2+k', \omega/2+\omega') G_0(k/2-k', \omega/2-\omega'). \quad (11)$$

Once again, Equation 11 is reproduced here from Hinrichsen [10]. The integral has been taken over d dimensions of space to prepare for the approximation $d = d_c - \epsilon = 4-\epsilon$ [15]. The integral is also only over the short wavelengths. Now the renormalized Green's function is included into the now rescaled action integral. This produces flow equations for p, D, and Γ. Letting the flow be stationary finds the critical points in coupling constant space, and the exponents fall out as

$$\chi = -2 + \frac{7\epsilon}{12}, \quad z = 2 - \frac{\epsilon}{12}. \quad (12)$$

Then the flow equations are linearized about the fixed point, and the eigenvalues of the linearized flow matrix yield the third independent exponent: $\nu_\perp = 1/2 + \epsilon/16 + \mathcal{O}(\epsilon^2)$. Combining this exponent with the definitions of χ and z, we find $\beta = 1 - \epsilon/6 + \mathcal{O}(\epsilon^2)$ and $\nu_\| = 1 + \epsilon/12 + \mathcal{O}(\epsilon^2)$ [8].

5 Experiments and Simulations

Allhoff and Eckhardt propose a slightly different percolation model in which there are *two* probabilistic parameters: r is the probability that an infected site spreads its infection to its nearest neighbors in the next time step. While p is the probability that a site continues to be infected in the next time step [12]. The direct simulations were performed in a 1+1 dimensional lattice. The authors hold p fixed at 0.7 and allow r to vary to a critical value, which they found to be about $r_c = 0.3096$. At $p = 0$, r_c shifted to 0.6447. However, the particular non-universal values of p and r did not change the universal behavior at the critical point, as expected. The authors report exponents of $\beta = 0.276$, $\nu_\perp = 1.087$, and $\nu_\| = 1.742$.

Because the spatial dimension is 1, $\epsilon = 3$, and recalling the results of the field theoretical calculations, $\beta = 1 - \epsilon/6 = 0.5$, $\nu_\perp = 1/2 + \epsilon/16 = 0.6875$, and $\nu_\| = 1+\epsilon/12 = 1.25$. Evidently, first order in ϵ isn't enough, and Ódor [8] cites the general $4-\epsilon$ expansion to second order, where $\beta = 0.39848$, $\nu_\perp = 0.8774$, and $\nu_\| = 1.45142$. These are getting warmer, but a low-density series expansion due to Jensen [17] provides exponent results that agree with Allhoff's experiment to within error bars.

This inaccuracy on the part of the ϵ-expansion is reasonable, because ϵ is so large in 1d directed percolation.

In physical turbulence experiments, these exponents are challenging to extract. However, Barkley has performed directed percolation simulations that replicate the characteristic puff and slug behavior of physical turbulence [18]. Also, Sipos [11] finds the mean lifetime of a turbulent puff in a 3+1 dimensional directed percolation model that agrees with an experiment due to Hof [19]. In a long cylindrical pipe, laminar flow was established by a constant pressure. A small perturbation upstream in the pipe sows the seed of turbulence, and a small puff is formed which moves downstream at the mean flow velocity. Laser Doppler anemometry was used to characterize the flow velocity and thus determine the lifetime of the puffs. Hof's results that the lifetime of puffs grows superexponentially as the critical Reynolds number is approached agrees with the simulations of Sipos [11].

6 Conclusion

The problem of fluid turbulence was posed, and a crude statistical model was developed in the universality class of directed percolation. The critical scaling exponents were naïvely calcualted with simple mean field theory, then expanded upon with the 4-ϵ expansion. Comparisons with simulation were made, but there are not enough physical experiments to verify these exponents, at least to my knowledge.

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