Renormalization Group for Interacting Fermions

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Abstract

The renormalization group approach of integrating out degrees of freedom successively has been crucial in understanding the critical phenomena. For bosonic case, this integration is done over momentum shells in the Wilsonian RG, but things are more complicated for fermions as their ground state consists of a Fermi surface around which the integrals are to be performed, as opposed to around a point for the bosonic case.

In this essay, we study the stability of a nonrelativistic fermionic system to interaction within a renormalization group framework, as discussed by R Shankar. The basic approach of RG here is analogous to that of integrating out on a momentum shells in a scalar field theory with a ϕ^4 interaction. The application of RG in 2 or 3 dimensions leads to Landau-Fermi liquid theory, with only relevant operators being those of BCS type (Cooper pair instability).

1 Introduction

The approach of renormalization group(RG), as developed by Kadanoff and Wilson in early 1970s, has enjoyed wide applicability in the theory of critical fluctuations in statistical mechanics, in quantum field theory and in condensed matter physics. The basic idea of RG involving rescaling the system and looking for the corresponding change in coupling constants has spawned a plethora of tools and techniques to compute the RG transformations for various systems.

In condensed matter physics, we deal with a unique system: a collection of large number of fermions ($N \sim 10^{23}$) on a lattice. For such a system, even at $T = 0$ there exists an energy level called the Fermi level upto which the energy levels are filled, owing to the Pauli's exclusion principle. Applying the ideas of renormalization there would involve integrating out high energy *excitations* with respect to the Fermi sea while keeping it intact. In this essay we demonstrate such calculations and the issues involved, as first done by R Shankar [1] back in early 1990s.

Physically, the verdict of interactions on Fermi sea was primarily deduced by the phenomenological Landau-Fermi liquid theory, which was initially proposed by Lev Landau to explain the behavior of ^{3}He at low temperatures. The theory is valid for repulsive interactions, while for attractive interactions, as $T \to 0$, we get the superconducting state as explained by the BCS theory. The renormalization group places the Fermi liquid theory on a solid footing by explaining how the interactions scale and hence affect the behavior of the system of interacting fermions. In this essay, we will demonstrate the Fermi liquid fixed point as well as the superconducting instability as obtained by RG calculations at $T = 0$.

The rest of this essay is organized as follows: §2 introduces the basic concepts relevant for the calculation, §3 describes the basic issues involved in RG for fermions, §4 goes over the RG calculation up to one loop in $d = 1$ and §5 continues that calculation for $d > 1$.

2 Setting the Stage

In this section, we set the stage for the forthcoming RG calculations for fermions. We discuss momentum shell RG, path integrals for fermions using Grassmann variables and basic idea of Landau-Fermi liquid theory.

2.1 Momentum Shell RG

The momentum shell RG consists of integrating out degrees of freedom corresponding to high energy modes and then rescaling momenta and fields accordingly [1,3]. The following schematic outline of procedure that will be used later for actual calculations in §4. To this end, consider an action $S\{\phi\}$ which is a function of some field ϕ, which we will assume to be bosonic for simplicity. The calculation of RG transformation consists of 3 steps:

1. Mode elimination: The first step is to split the fields (in momentum space) among *fast* and *slow* modes, $\phi_<(\mathbf{k})$ and $\phi_>(\mathbf{k})$, respectively, w.r.t. a sharp cutoff.

$$\phi_<(\mathbf{k}) = \phi(\mathbf{k}) \; ; \quad 0 < k < \Lambda/s$$
$$\phi_>(\mathbf{k}) = \phi(\mathbf{k}) \; ; \quad \Lambda/s < k < \Lambda$$

where $s = e^t > 1 \Rightarrow t > 0$ defines the momentum shell which we are integrating out. Substituting this in action will lead to

$$S\{\phi(\mathbf{k})\} = S_<\{\phi_<(\mathbf{k})\} + S_>\{\phi_>(\mathbf{k})\} + S_I\{\phi_<(\mathbf{k}), \phi_>(\mathbf{k})\}$$

where the last term mixes the modes at low and high momenta. Integrating out fast modes and using cluster expansion results in a Wilsonian effective action of the form [1, 3]

$$S'_< = S_< + \langle S_I \rangle + \frac{1}{2}\left[\langle S_I^2 \rangle - \langle S_I \rangle^2\right] + \ldots$$

2. Rescaling momenta: To compare S' with S, we want the new field theory to be defined with a cutoff Λ (and not Λ/s). Hence, we rescale all momenta in the new action as $k \to k' = sk$.

3. Rescaling fields: Finally, the new action $S'_<$ can such that the new coupling constants are simply scaled versions of the old ones. Hence, we choose one term (generally the kinetic term) whose coefficient we want to keep fixed and then see the variation of other coupling constants w.r.t. that one. We can rescale the fields accordingly to get sensible coupling constants.

These steps constitute one iteration of the RG transformation. The final result is a perturbative calculation of the renormalized parameters as a function of the original parameters and s, the renormalization scale. Once we have these RG transforms, we can compute the β-function and the RG fixed points(writing $s = 1+t$, $t \ll 1$ and differentiating w.r.t. t), thereby calculating the flows and understanding the phase diagram of the system.

2.2 Fermionic path integrals

As the fermionic operators anticommute, we need anticommuting fields to represent them. To this end, we define Grassmann numbers ψ and $\bar{\psi}$, which anticommute amongst themselves as well as with fermionic fields. Clearly, we need a negative sign whenever two Grassmann fields are exchanged, but no extra sign when pairs of them are moved around. Using these fields, the path integral is defined as

$$\mathcal{Z} = \int [d\bar{\psi} d\psi] e^{S_0} \, , \quad S_0 = \int d^d\mathbf{r} \int_0^\beta d\tau \bar{\psi}(\mathbf{r}, \tau) \left(\partial_\tau - H(\mathbf{r})\right) \psi(\mathbf{r}, \tau) \tag{1}$$

For zero temperature calculations, $\beta \to \infty$. Also, it's convenient to transform the fields to the Fourier $(\mathbf{K}, \omega)$ domain, leading to an action

$$S_0 = \int \frac{d^d\mathbf{K}}{(2\pi)^d} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, \bar{\psi}(\mathbf{K}, \omega) \left(i\omega - H(\mathbf{K})\right) \psi(\mathbf{K}, \omega) \tag{2}$$

This is the form of action that we will use in the subsequent sections.

2.3 Landau-Fermi liquid theory

The basic claim of the Landau-Fermi liquid theory is that as we *adiabatically* turn on repulsive interactions between fermions in a fermi gas, the excitations for the interacting system are in one-to-one correspondence with those of the noninteracting system [4]. The excitations in the interacting theory are fermionic *quasiparticles*, with a renormalized mass m^* (which is generally taken as a phenomenological parameter) and a finite lifetime which diverges as one approaches the Fermi level. Hence, if one only cares about the excitations close to the Fermi level, then the claim is that the system with repulsive interactions behaves identical to a noninteracting system with a renormalized mass($m \to m^*$).

3 RG for fermions : Central issues

Consider a bunch of spinless fermions on a lattice, a system (assuming spin to be irrelevant, which is true if the spin degeneracy is not lifted) quite typical in condensed matter physics. The ground state will have one-particle states occupied upto certain energy μ, the Fermi energy, which corresponds to the *Fermi surface* defined for $E(\mathbf{K}) = \mu$. This noninteracting ground state has gapless excitations – electrons can be excited to levels arbitrarily close to the Fermi level.

The basic problem is to understand if that stays true when we turn on the interactions (or if the system develops a gap instead). Landau Fermi liquid theory, as described earlier, phenomenologically claims that there should be no gap for any repulsive interaction. To understand why that might be true, we can seek a solution using RG by looking for relevant operators near the noninteracting *fixed point*: if there are any such operators, the system will flow away from the gapless state and will develop a gap.

To construct a RG transformation in this case, we need to measure energies w.r.t the Fermi level and integrate out the high energy excitations. In terms of $\mathbf{K}$, momenta measured from the origin, these states will lie in two shells, one outside and one inside the Fermi surface, with radii $K_F \pm \Lambda$. Also, even when we are done integrating out high energy modes, unlike the bosonic case, we should still be left with scatterings with momentum transfer of order $2K_F$, pertaining to fermions scattering on the Fermi surface. Hence, one needs to integrate out only those momenta which exceed Λ from the Fermi surface in the radial directions, while keeping intact the momenta in the angular directions. This is the central issue which is addressed in this essay.

4 RG in $d = 1$

We start with the $d = 1$ (1 spatial dimension) case for fermions as it is particularly analogous to the bosonic case as the Fermi *surface* in $d = 1$ consists of just two points $K_F = \pm\pi/2$. In this section we will construct RG flows for the fermionic action based on the procedure outlined in §2.2, to serve as a template for the more complicated calculations for $d > 1$.

4.1 Hamiltonian and basic considerations

The Hamiltonian on a lattice with nearest neighbor repulsion is

$$\mathcal{H} = -\frac{1}{2}\sum_n \psi^\dagger(n+1)\psi(n) + U_0 \sum_n (n(i) - \frac{1}{2})(n(i+1) - \frac{1}{2}) + \text{h.c.} \tag{3}$$

where $n = \psi^\dagger\psi$ is the number operator at each site. In this Hamiltonian, the chemical potential is $\mu = U_0$, which maintains half filling. Now, for U_0 the system is completely delocalized owing to the hopping term, while for $U_0 \to \infty$, the system develops a *charge density wave*(CDW) order with alternate sites occupied as it prefers alternating $n = 0$ and $n = 1$ sites to minimize the interaction term. The former is a gapless state, but the latter (CDW) is gapped as it costs a finite energy to occupy an empty site.

The central question is whether a CDW order develops for any value of $U_0 > 0$. Mean field theory tells is that it is so, that the system develops a gap for any positive value of U_0 given by an equation analogous to the *superconducting gap equation*

$$1 = U_0 \int_0^\pi \frac{dK}{\pi} \frac{1}{\sqrt{E^2(K) + \Delta^2 U_0^2}} \quad \Rightarrow \quad \Delta \sim \frac{\Lambda}{U_0} e^{-\pi/2U_0} \; ; \quad \Lambda \equiv \text{cutoff}$$

But an exact solution of this model by Yang tells us that there is no CDW order until $U_0 \sim 1$. Evidently the mean field theory breaks down in this case and we can't ignore fluctuations here. We now perform RG calculations on this model to see if we can get any better results. To this end, we Fourier transform $\psi(i)$ and expand around the Fermi level $K_F = \pi/2$ as $E(K) = cos(K) = cos(\pi/2 + k) \approx k$. Here, k (or $\mathbf{k}$ for $d > 1$) is measured from the Fermi level as opposed to K (or $\mathbf{K}$ for $d > 1$) measured from the origin, a notation that we will follow throughout this essay.

4.2 RG calculation

The *free* Hamiltonian, expanded in the vicinity of Fermi level is

$$\mathcal{H}_0 = \sum_i \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \psi_i^\dagger(k)\, k\, \psi_i(k) \; ; \quad k = K - K_F, \; i = L, R \tag{4}$$

for which the path integral becomes (§2.2)

$$\mathcal{Z}_0 = \int \prod_{i=L,R} \prod_{|k|<\Lambda} d\bar{\psi}_i(\omega, k) d\psi_i(\omega, k) e^{S_0(\psi, \bar{\psi})} \tag{5}$$

with

$$S_0 = \sum_{i=L,R} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}_i(\omega, k)(i\omega - k)\psi_i(\omega, k) \tag{6}$$

In the first step of RG, we want to integrate out all degrees of freedom in the shell $\Lambda/s \le |k| \le \Lambda \; , s > 1$ which corresponds to two regions $\Lambda/s \le k \le \Lambda$ and $-\Lambda \le k \le -\Lambda/s$.

For the free action S_0, integrating out $\psi_>$ gives a constant as there is no $S_I(\psi_<, \psi_>)$ term mixing the slow and fast modes. Ignoring that constant addition to the new action,

$S'_<(\bar{\psi},\psi) = S(\bar{\psi},\psi)|_{\Lambda\to\Lambda/s}$. Now we rescale the frequency and momenta (step 2, §2.1) as $k' = sk, \omega' = s\omega$ to take the cutoff back to Λ, resulting in

$$S'_<(\bar{\psi},\psi) = s^3 \sum_{i=L,R} \int_{-\Lambda}^{\Lambda} \frac{dk'}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \bar{\psi}_i(\omega'/s, k'/s)(i\omega' - k')\psi_i(\omega'/s, k'/s) \tag{7}$$

We can now redefine the fields (step 3, §2.1) as

$$\psi_i(\omega'/s, k'/s) = s^{-3/2}\psi'_i(\omega', k')\,, \quad \bar{\psi}_i(\omega'/s, k'/s) = s^{-3/2}\bar{\psi}'_i(\omega', k')$$

which results in

$$S'(\bar{\psi}',\psi') = \sum_{i=L,R} \int_{-\Lambda}^{\Lambda} \frac{dk'}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \bar{\psi}'_i(\omega', k')(i\omega' - k')\psi'_i(\omega', k') = S(\bar{\psi}',\psi') \tag{8}$$

Hence, the free action we started off with is the Gaussian fixed point of this system. We will compute the relevance/irrelevance of operators with respect to this fixed point.

Quadratic perturbations: For any quadratic perturbation of the form

$$\delta S_2 = \sum_{i=L,R} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}_i(\omega, k)\mu_i(\omega, k)\psi_i(\omega, k) \tag{9}$$

the renormalization proceeds exactly as the case of the free action. On rescaling the momenta and the fields, we find that

$$\mu'_i(\omega', k') = s\mu_i(\omega'/s, k'/s) \tag{10}$$

The simplest way to see this is to notice the fact that $\mu(\omega, k)$ simply replaces $(i\omega - k)$ in the free action S_0, which spits out a factor of s on rescaling. Hence, for consistency, μ should do the same. Expanding both sides of the scaling relation for μ (eqn 10) in powers of k, ω (and ignoring the left/right index i) give

$$\mu(\omega, k) = \mu_{00} + \mu_{10} i\omega + \mu_{01} k + \cdots + \mu_{mn}(i\omega)^m k^n + \ldots$$

which leads to the scaling for coefficients

$$\mu'_{00} = s\mu_{00}\,, \quad \mu'_{10} = \mu_{10}\,, \quad \mu'_{01} = \mu_{01}\,, \quad \mu'_{20} = s^{-1}\mu_{20} \text{ etc.} \tag{11}$$

Hence, μ_{00} is relevant (increases with s), μ_{10} and μ_{01} are marginal and all higher couplings are irrelevant, as they scale as s^{-1} or smaller powers of s.

Now the constant coupling μ_{00} in ($\mu_{00}\bar{\psi}\psi$ term) corresponds to a change in the chemical potential which is used to define the Fermi level in the first place. The Fermi level is defined as the the energy where $E(K) = \mu$; any μ_{00} should be absorbed in the definition of the Fermi level. Hence, at tree level, $\mu_{00} = 0$. (We will see that there are one-loop corrections from interactions.) As for the other two terms, they simply rescale the coefficients in

the terms we already have in the free action, and therefore do not count for any *interaction*. Hence, the quadratic perturbation doesn't give us anything new under RG transform.

Quartic interaction: Consider general quartic interactions of the form

$$\delta S_4 = \frac{1}{2!2!}\int_{K\omega} \bar{\psi}(4)\bar{\psi}(3)\psi(2)\psi(1)u(4321)\ ; \quad \psi(i) = \psi(\omega_i, K_i) \tag{12}$$

where the integration measure is

$$\int_{K\omega} = \left[\prod_{i=1}^{4}\int_{-\pi}^{\pi}\frac{dK_i}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega_i}{2\pi}\right] 2\pi\bar{\delta}(K_1+K_2-K_3-K_4)2\pi\delta(\omega_1+\omega_2-\omega_3-\omega_4)$$

where $\bar{\delta}(K)$ refers to a delta function modulo 2π to account for the *umklapp* processes. The coupling function $u(4321)$ stands for $u_{i_4 i_3 i_2 i_1}(\omega_i, k_i)$, where the i_n's label the sector L, R of the nth fermion. Here, the momenta K_i's are measured from the origin and not from the Fermi level. In this action, indices 1 and 2 correspond to incoming particles and 3 and 4 to outgoing ones. The coupling function u is supposed to obey all the symmetries of the problem. Here, as ψ's are Grassmann which anticommute, u should also pick up a sign under exchange of the two incoming or the two outgoing particles, as

$$u(4321) = -u(3421) = -u(4312) = +u(3412) \quad \forall\ k, \omega$$

At tree level contribution which renormalizes u, all ψ's in δS_4 are slow modes and integrating over fast modes simply amounts to integrating constants, which is again a constant and can be discarded. In order to scale the momenta, we need to compute the scaling for delta function. For frequencies, it is simply

$$\delta(\omega_1'+\omega_2'-\omega_3'-\omega_4') = s\delta(\omega_1'/s+\omega_2'/s-\omega_3'/s-\omega_4'/s)$$

but things are more complicated for the momentum $\bar{\delta}$ function as what we are scaling is momentum w.r.t the Fermi surface, k, which is defined by the relation

$$K = \varepsilon_i(K_F + k)\,, i = L, R\ ; \quad \varepsilon_L = -1,\ \varepsilon_R = +1$$

which when substituted in $\bar{\delta}$ gives

$$\bar{\delta}(K_1+K_2-K_3-K_4) = \bar{\delta}(K_F\sum_i \varepsilon_i + \sum_i \varepsilon_i k_i) = \bar{\delta}(\sum_i \varepsilon_i k_i)$$

Now, conservation of momentum demands that the only interactions allowed are of type $LL, RR \to LL, RR$ and $LR \to RL$. This is because an odd number of R's (and hence L's) would imply momentum imbalance of order K_F. Restricting to these interactions gives $\sum_i \varepsilon_i = 0, \pm 4 \Rightarrow K_F\sum_i \varepsilon_i = 0, \pm 2\pi$. Hence the first term drops out of the periodic delta function and everything depends only on k's, leading to

$$\bar{\delta}(k_1'+k_2'-k_3'-k_4') = s\bar{\delta}(k_1'/s+k_2'/s-k_3'/s-k_4'/s)$$

The individual measures simply transform to $dk = dk'/s$, where $dk = \varepsilon_i dK$ and $\prod_i \varepsilon_i = 1$, as the $\varepsilon_L = -1$'s always come in pairs. Hence the total integration measure transforms by a factor of $(s^{-2})^4 s^2 = s^{-6}$. Scaling the fields as $\psi = s^{-3/2}\psi'$ leads to

$$u'_{i_4 i_3 i_2 i_1}(\omega'_i, k'_i) = u_{i_4 i_3 i_2 i_1}(\omega'_i/s, k'_i/s) \tag{13}$$

Expanding in Taylor series as in the quadratic case, we can see that all couplings except the constant term is irrelevant. The marginal term results in

$$u_0 = u_{LRLR} = u_{RLRL} = -u_{RLLR} = -u_{LRRL} \tag{14}$$

Here, only the interactions of form $LR \to RL$ are considered as the others are eliminated by the requirement of antisymmetry of fermions. This is because as there is no k dependence here (only the constant term in $u(k, \omega)$ counts), this would lead to two fermions being scattered to the same state, which is forbidden by the Pauli's exclusion principle. Hence, the only surviving term is a marginal coupling constant u_0 at tree level.

One loop corrections to μ: The only contributing diagram for μ is the tadpole (Fig. 1(a)). We can set both the external momenta k and frequency ω to zero, and assume them to lie at L. The 1/2!2! factor in the interaction is cancelled by the diagram combinatorics, leading to

$$\mu' = s\left[\mu - u_0 \int_{\Lambda/s<|k|<\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega 0^+}}{i\omega - k}\right] \tag{15}$$

We need a factor of $e^{i\omega 0^+}$ to make the ω integral converge, which now can be performed as a contour integral. The contour needs to be closed in the upper half plane ($\omega \to i\infty$) and there is a pole at $\omega = -ik$. The result is

$$\mu' = s\left[\mu - u_0 \int_{d\Lambda} \frac{dk}{2\pi}\Theta(-k)\right] = s\left[\mu - \frac{u_0}{2\pi}\int_{-\Lambda}^{-\Lambda/s} dk\right] = s\left[\mu - \frac{u_0\Lambda}{2\pi}\left(1 - \frac{1}{s}\right)\right] \tag{16}$$

The RG flow can be obtained by substituting $s = 1+t$, expanding to order t and computing the derivative w.r.t. t

$$\mu' = (1+t)\left[\mu - \frac{u_0\Lambda}{2\pi}t\right] = \mu + t\left[\mu - \frac{u_0\Lambda}{2\pi}\right] \Rightarrow \frac{d\mu}{dt} = \mu - \frac{u_0}{2\pi} \tag{17}$$

The fixed point is given by $\mu' = \mu = \mu^*$. Substituting and solving,

$$\mu^* = s\left[\mu^* - \frac{u_0^*\Lambda}{2\pi}\left(1 - \frac{1}{s}\right)\right] = s\left[\mu^* - \frac{u_0^*\Lambda}{2\pi}\right] + \frac{u_0^*\Lambda}{2\pi} \Rightarrow \mu^* = \frac{u_0^*\Lambda}{2\pi} \tag{18}$$

One loop corrections to u_0: For this correction, we will need to go over to $\langle S_I^2\rangle - \langle S_I\rangle^2$ term in the cluster expansion, where the second term cancels out all Feynman diagrams which contain disconnected pieces [3] and we only need to worry about the diagrams which are completely connected. Now we only need to bother about 3 diagrams (or channels),

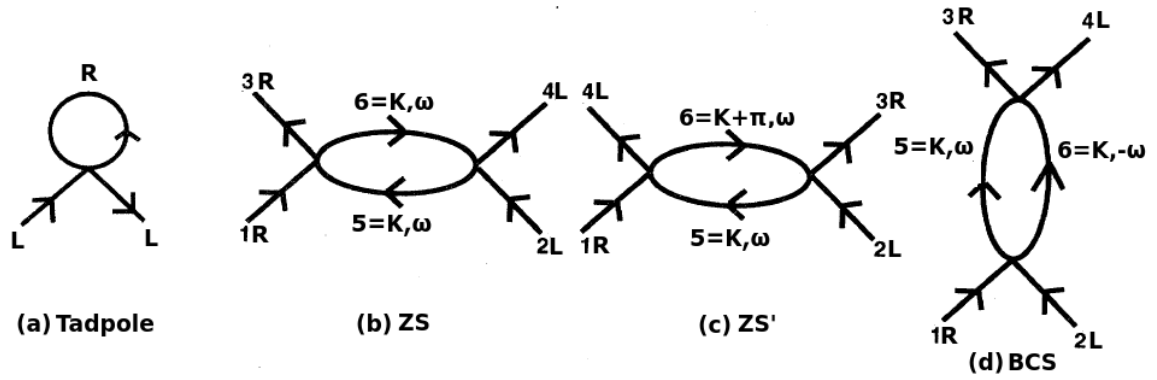

Figure 1: Feynman diagrams for corrections at one loop for (a) μ and (b)-(d) u_0. Each vertex consists of 2 L's and 2 R's. For the tadpole, the combinatorial factor is $2 \times 2 = 4$ (2 L's and 2 R's) which cancels 2!2! in the coupling constant. For ZS and ZS', each vertex is analogus to the tadpole, leading to a factor of $(2!2!)^2$ which is cancelled by the denominator of $(u_0/2!2!)^2$. For BCS, there are only 2 ways to connect legs 5 and 6 (as opposed to 4 in ZS and ZS'), leading to an overall factor of $2(2!)^2$.

generally labelled as ZS, ZS' and BCS, as shown in Fig. 1. Let the change in u_0 be denoted by δu_0. This change is schematically given by

$$\begin{aligned}\delta u_0(4321) = &\int u(6351)u(4526)G(5)G(6)\delta(3+6-1-5)d5d6 \qquad \text{(ZS)}\\ &-\int u(6451)u(3526)G(5)G(6)\delta(6+4-1-5)d5d6 \qquad \text{(ZS')}\\ &-\frac{1}{2}\int u(6521)u(4365)G(5)G(6)\delta(5+6-1-2)d5d6 \quad \text{(BCS)} \qquad (19)\end{aligned}$$

where the factor of $1/2$ in front of the BCS term is from combinatorics, as explained in the figure caption. The signs are due to fermion exchanges (fermionic Wick's theorem). For instance, the L and R are exchanged between 3 and 4 in ZS and ZS', leading to a relative minus sign. Here 1-4 stand for the slow modes with $(4321) \equiv (LRLR)$ and 5-6 are the fast modes which are integrated over, The G's stand for Green's functions (propagators) and $d5$ stands for $dk_5 d\omega_5$ etc. Now the external frequencies and momenta can be set to zero to perform the integrals.

For ZS, there is no momentum transfer at the vertices, as $L \to L$ and $R \to R$, hence the momentum K is same for $G(5)$ and $G(6)$. On the other hand, for ZS' and BCS, there is a momentum transfer of π, as $L \to R$ and $R \to L$, leading to one of the momenta being $K' = K + \pi$. As far as frequencies are concerned, they are same in both legs for ZS and ZS' but opposite for BCS, owing to the direction of flow as depicted in the diagram. Also we know that $E(-K) = E(K)$ and $E(K' = K + \pi) = -E(K)$. Using all this, the expression for δu_0 for zero external momenta can be expressed as

$$\begin{aligned}\delta u_0(4321) = &\int_{d\Lambda} \frac{dK}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{u(KRKR)u(LKLK)}{[i\omega - E(K)][i\omega - E(K)]} - \int_{d\Lambda} \frac{dK}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{u(K'LKR)u(RKLK')}{[i\omega - E(K)][i\omega + E(K)]}\\ &-\frac{1}{2}\int_{d\Lambda} \frac{dK}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{u((-K)KLR)u(LR(-K)K)}{[i\omega - E(K)][-i\omega - E(K)]} \qquad (20)\end{aligned}$$

Now we just need to evaluate this. Again, we can do the ω integral as a contour integral, where the contour can be closed in either top or bottom plane owing to the $1/\omega^2$ decay of the argument. The calculations are as follows:
ZS: In this case, the two poles coincide at $\omega = iE(K)$. The contour can always be closed in such a way that it doesn't include these poles, and hence ZS $= 0$.
ZS': In this case, the poles lie at $\omega = \pm iE(K) = \pm i|E(K)|$. Hence, for any choice of contour, it encircles one of these poles. Also, if K lies near L, then the coupling constants are $u(K'LKR) = u(RLLR) = -u_0$ and $u(RKLK') = u(RLLR) = -u_0$. Using the residue at that pole and the fact that near the Fermi surface, $E(K) \approx |k| \approx \Lambda$ gives

$$ZS' = u_0^2 \int_{d\Lambda \in L} \frac{dK}{2\pi} \frac{1}{2|E(k)|} = \frac{u_0^2}{4\pi\Lambda} \int_{d\Lambda \in L} dK = \frac{u_0^2}{4\pi\Lambda} 2d\Lambda = \frac{u_0^2}{2\pi} \frac{d\Lambda}{\Lambda}$$

BCS: Similar to ZS', the poles here lie at $\omega = \pm iE(K) = \pm i|E(K)|$. But in this case K is allowed to lie in either L or R, which cancels out the factor of 2 before this term. Also, there is a overall negative sign compared to ZS' due to the $-i\omega - E(K)$ term in BCS. Hence the final contribution is

$$BCS = -ZS' = -\frac{u_0^2}{2\pi} \frac{d\Lambda}{\Lambda}$$

Finally, summing them up,

$$\delta u_0 = ZS - ZS' - BCS = 0 - \frac{u_0^2}{2\pi} \frac{d\Lambda}{\Lambda} + \frac{u_0^2}{2\pi} \frac{d\Lambda}{\Lambda} = 0 \tag{21}$$

Hence, up to one loop order, there is no renormalization of the coupling constant u_0.

4.3 Interpreting the results

The final result for RG flows up to one loop are

$$\frac{d\mu}{dt} = \mu - \frac{u_0}{2\pi} \; ; \quad \frac{du_0}{dt} = 0 \tag{22}$$

which leads to a line of fixed points

$$\mu^* = \frac{u_0^*}{2\pi} \, , \quad u_0^* = \text{ arbitrary} \tag{23}$$

Physically, this refers to the fact that we need to make corrections to the Fermi level as we increase the interaction strength in order to maintain the same density of fermions.

Hence, RG tells us that there is no gap for small values of u_0, in agreement with the exact solution. The exact result is reporduced as we increase the coupling u_0 as eventually the $RR \to LL$ *umklapp* coupling, which is irrelevant at Gaussian fixed point, becomes relevant and makes the system flow towards CDW fixed point [1].

5 RG in $d > 1$

For $d > 1$, the Fermi surface is actually a $d-1$ dimensional surface about which we need to compute the RG transform. This will lead to some new phenomena which are not present in $d = 1$. We start off with $d = 2$, where the Fermi surface is a circle. For the energy, we would use the free particle spectrum, which corresponds to the lattice spectrum for small momenta, and write

$$\varepsilon(K) = E(K) - \mu = \frac{K^2 - K_F^2}{2m} = \frac{K + K_F}{2m}(K - K_F) = \frac{K_F}{m}k = v_F k \tag{24}$$

where $K \sim K_F$ and v_F is the Fermi velocity. The free fermion action is

$$S_0 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \bar{\psi}(\omega, \theta, k)(i\omega - v_F k)\psi(\omega, \theta, k) \tag{25}$$

which is essentially analogous to the $d = 1$ case, except for the fact that the ψ's are now labelled by a continuous $\theta \in [0, 2\pi)$ instead of the discrete labels $i \in \{L, R\}$. The *trick* to get this is to write the measure d^2K as $d^2K = KdKd\theta = K_F dk d\theta$ and absorb the factor of K_F in the fields as $\psi \to \sqrt{K_F}\psi$, $\bar{\psi} \to \sqrt{K_F}\bar{\psi}$.

Tree level RG: As this is identical to the $d = 1$ problem with an extra label θ, the quadratic interactions calculation proceeds as it was done previously, with the terms either modifying the chemical potential (μ_{00}) or the existing terms (μ_{01}, μ_{10}). The quartic interactions give qualitatively similar result at tree level, but there is an additional subtlety involved, which we will now discuss.

In $d = 2$, as the momenta point anywhere on the Fermi surface, we need an extra term to make sure that all momenta lie within the shell. Hence, the integration measure becomes

$$\int_{K\omega\theta} = \left[\prod_{i=1}^{3} \int_{-\infty}^{\infty} \frac{d\omega_i}{2\pi} \int_0^{2\pi} \frac{d\theta_i}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dk_i}{2\pi}\right] \Theta(\Lambda - |k_4|)\ ; \quad k_4 = |\mathbf{K}_4| - K_F$$

where the step function $\Theta(\Lambda - |k_4|)$ ensures that after choosing k_i, $i = 1, 2, 3$ arbitratily, k_4 does not lie beyond the shell of thickness Λ. Now, the issue is that under the scaling of k_i's, there is no simple scaling for $\Theta(\Lambda - |k_4|)$ here. In fact,

$$\Theta(\Lambda - |k_4(k_1, k_2, k_3, K_F|) \to \Theta(\Lambda - |k_4'(k_1', k_2', k_3', sK_F|)$$

The way out of this is to replace the hard cutoff Θ with a soft exponentially decaying cutoff, via $\Theta(\Lambda - |k_4|) \to e^{-|k_4|/\Lambda}$. Assuming that the direction of $\mathbf{K}_i$ is given by $\mathbf{\Omega}_i$, $|k_4|$ can be written as

$$k_4 = |K_F(\mathbf{\Omega}_1 + \mathbf{\Omega}_2 - \mathbf{\Omega}_3) + k_1\mathbf{\Omega}_1 + k_2\mathbf{\Omega}_2 + k_3\mathbf{\Omega}_3| - K_F \simeq K_F(|\mathbf{\Delta}| - 1)$$

where we define $\mathbf{\Delta} = \mathbf{\Omega}_1 + \mathbf{\Omega}_2 - \mathbf{\Omega}_3$ and ignore the k_i term as $k_i \ll K_F$. Now, scaling as in the case of $d = 1$,

$$u'(k', \omega', \theta)e^{-sK_F(|\mathbf{\Delta}|-1)/\Lambda} = u(k'/s, \omega'/s, \theta)e^{-K_F(|\mathbf{\Delta}|-1)/\Lambda}$$

leading to

$$u'(k',\omega',\theta) = u(k'/s,\omega'/s,\theta)e^{-K_F(|\mathbf{\Delta}|-1)(s-1)/\Lambda} \tag{26}$$

Now, only those couplings survive for which the exponential prefactor is equal to 1 as everything else scales to 0. This leads to the condition for marginal operators as

$$|\mathbf{\Delta}| = |\mathbf{\Omega}_1 + \mathbf{\Omega}_2 - \mathbf{\Omega}_3| = 1 \tag{27}$$

which in $d = 2$ has three solutions ($|\mathbf{\Omega}_i| = 1$ as $\mathbf{\Omega}_i$'s are unit vectors. Also, $\sum_i \mathbf{\Omega}_i = 0$)

- $\mathbf{\Omega}_1 = \mathbf{\Omega}_3 \Rightarrow \mathbf{\Omega}_2 = \mathbf{\Omega}_4$: (3,4) slaved to (1,2)
- $\mathbf{\Omega}_2 = \mathbf{\Omega}_3 \Rightarrow \mathbf{\Omega}_1 = \mathbf{\Omega}_4$: (3,4) slaved to (1,2)
- $\mathbf{\Omega}_1 = -\mathbf{\Omega}_2 \Rightarrow \mathbf{\Omega}_3 = -\mathbf{\Omega}_4$: (2,4) slaved to (1,3)

The first two conditions imply that the incoming momenta are individually conserved. The third case is nontrivial: when the incoming momenta are equal and opposite, the final momenta can take any pair of values on the Fermi surface which are equal and opposite.

Hence, unlike $d = 1$, the 4-point coupling is described by two coupling constants:

$$\begin{aligned} F(\theta_{12}) =& F(\theta_1 - \theta_2) = u(\theta_1 = \theta_3,\, \theta_2 = \theta_4) = -u(\theta_1 = \theta_4,\, \theta_2 = \theta_3) \\ V(\theta_{13}) =& V(\theta_1 - \theta_3) = u(\theta_1 = \theta_2,\, \theta_3 = \theta_4) \end{aligned} \tag{28}$$

where F correspond to the first two cases as they are related by an exchange $3 \leftrightarrow 4$. Both F and V are rotation invariant. Here, F is symmetric in its argument while V is antisymmetric, owing to the antisymmetry of fermions.

In $d = 3$, the analysis proceeds exactly as $d = 2$, except that we have an additional angle ϕ. Hence, at tree level,

$$F(\theta_{12}) \to F(\theta_{12}, \phi_{12;34})\,, \quad V(\theta_{13}) \to V(\theta_{13})$$

where V doesn't need a ϕ as it's rotationally symmetric about the initial direction θ_1.

One loop corrections: For μ, as in $d = 1$, only tadpole diagram contributes and only $F(\theta_{12})$ is needed as we set both the external legs to zero, hence one of the incoming angles needs to be same as one of the outgoing ones. The fixed point is

$$\delta\mu^* = \int \frac{d\omega dk d\theta}{(2\pi)^3} \frac{F(\theta - \theta')}{i\omega - v^* k} = \int_{-\Lambda}^{0} \frac{dk}{2\pi} \int_0^{2\pi} \frac{d\theta'}{2\pi} F(\theta - \theta') = -\frac{\Lambda}{2\pi} \int_0^{2\pi} \frac{d\theta'}{2\pi} F(\theta') \tag{29}$$

which is a constant. As for renormalization of the quartic couplings, we again have those 3 Feynman diagrams ZS, ZS' and BCS, summing up to

$$\begin{aligned} \delta F(4321) = & \int_{d\Lambda} \frac{dK}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{u(K+Q,3,K,1)u(4,K,2,K+Q)}{[i\omega - E(K)][i\omega - E(K+Q)]} \\ & - \int_{d\Lambda} \frac{dK}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{u(K+Q',4,K,1)u(3,K,2,K+Q')}{[i\omega - E(K)][i\omega + E(K+Q')]} \\ & - \frac{1}{2} \int_{d\Lambda} \frac{dK}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{u(P-K,K,2,1)u(4,3,P-K,K)}{[i\omega - E(K)][-i\omega - E(P-K)]} \end{aligned} \tag{30}$$

where $Q = K_3 - K_1$, $Q' = K_4 - K_1$ and $P = K_1 + K_2$ in the ZS, ZS' and BCS diagrams, respectively. It is implicit that the momenta are all in $d\Lambda$. Now, we need to choose the variables in $u()$'s such that we get F or V interactions. As far as F is concerned, all three diagrams vanish. The vanishing of ZS is analogus to the $d = 1$ case and ZS' here is same as ZS. For BCS, requiring that both K and $P - K$ lie in a shell $d\Lambda$ leads to contribution of order $d\Lambda^2 \equiv 0$ at this order.

For V, the ZS and ZS' diagrams vanish again (as for F), but BCS contributes as

$$\frac{dV(\theta_{13})}{dt} = -\frac{1}{8\pi^2}\int_0^{2\pi}\frac{d\theta}{2\pi}V(\theta_1 - \theta)V(\theta - \theta_3) \tag{31}$$

Switching over to Fourier space for $V(\theta)$,

$$V_n = \int_0^{2\pi}\frac{d\theta}{2\pi}e^{in\theta}V(\theta) \Rightarrow \frac{dV_n}{dt} = -\frac{V_n^2}{4\pi} \tag{32}$$

Hence, this coupling is irrelevant for $V_n > 0$ (repulsive interaction) and relevant for $V_n < 0$ (attractive interaction). Similar expressions are obtained for $d = 3$. This is exactly what we expect from the Landau-Fermi liquid theory: Any repulsive interaction scales away and we flow back to the noninteracting fixed point, while an attractive interaction, however small, leads to the system flowing away from the Gaussian fixed point. This is the superconducting instability, which leads to opening up of a gap at the Fermi surface.

To summarize, RG tells us that in $d > 1$, a system with repulsive interactions is identical to a noninteracting system with renormalized parameters (Landau-Fermi liquid theory) while a system with attractive interactions eventually opens up a gap at the Fermi level (superconducting instability).

6 Conclusion

This essay demonstrates the renormalization group transformation calculation for fermions about a Fermi surface. In $d = 1$, RG gives the result in harmony with the exact solution and identifies the problem with mean field theory. For $d > 1$, we deduce the Landau-Fermi liquid theory for any repulsive interaction as well as obtain the superconducting instability for an attractive interaction. Hence, starting from a Fermi gas, RG reproduces all the features of weakly interacting fermions.

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