Critical Phenomena in Stationary Black Hole Spacetimes

Suraj S Hegde

Dept. Of Physics,University of Illinois,Urbana-Champaign

(Dated: May 17, 2013)

Abstract

This essay aims to be a study in the phase transitions and critical behavior aspects of stationary black hole space times, like Kerr-Newman black holes.Though there are divergences in the thermodynamic susceptibilities of the black holes,which might signal a phase transition,there are are certain intricacies involved in their interpretation.Using the thermodynamic fluctuation theory it is seen that divergences at a continuous turning point of the thermodynamic function need not be a critical point but only indicates a change in stability.This gives us some insight into the relation between fluctuations and critical phenomena.It is found that the extremal phase of the Kerr-Newman black hole corresponds to a critical point of a continuous phase transition.One can also obtain the critical exponents and scaling laws at that point. Also the formalism of fluctuation theory can be extended to give a geometric interpretation of some of the usual critical phenomena in regular systems. In the end,just a mention is made of a recent development in considering the entanglement entropy of the black hole under the framework of Renormalisation Group.

1. INTRODUCTION

"Black Holes of nature are the most perfect macroscopic objects that are in the universe;The only elements in their construction are are our concepts of space and time.And since the general theory of relativity gives only a single unique family of solutions for their descriptions,they are the simplest objects as well"-This is a very beautiful description of black holes given by S.Chandrashekar in his book "The Mathematical Theory of Black Holes".But it was a very shocking revelation of Hawking,that black holes emit radiation,while they were considered only to absorb objects.It was even more shocking when Beckenstein showed that the temperature of the radiation can be attributed as the temperature of the black hole itself.While the fact that black holes are described by a set of very minimal number of fundamental quantities like mass,spin and charge,it is very puzzling that it be attributed with a thermodynamical concept of temperature.And we know that the very reason we have thermodynamical description of matter,is due to the macroscopic constituency.This puzzling contrast in the behavior of black holes stays unexplained even today,except for the proposed solutions of String theory.

On the other hand ,the mere fact that one can attribute equilibrium thermodynamical functions to black holes has tempted many to proceed further and study fluctuations and phase transitions related phenomena in them.They have also found critical behavior, like scaling power laws,divergence of fluctuations etc. But there are certain intricacies in interpreting these notions in the context of black holes.Also we do not have a complete microscopic picture of black holes as of now.This being the case there have been attempts to apply methods of renormalisation group to the entanglement entropy of the black holes,which have been found to contribute in part to the entropy of Hawking radiation.

2. BLACK HOLE THERMODYNAMICS

If we consider a Kerr-Newman black hole,one can extract energy from it by sending in particles with angular momentum,opposite to that of black hole.This is called the Penrose process.One can ask to what end can this process be continued.As the Penrose process is possible only with the existence of an ergosphere(which is the spatial limit of existence for a static observer),the point at which it tends to become a Schwarzschild black hole,would mark the end of extraction.One can show that the constraints imposed on the variation of the parameters of the black hole during the extraction will lead to the following condition on the area of the horizon:

$$dA \geq 0 \tag{1}$$

The equation relating the area to the variation of other parameters is given by

$$dm = \frac{\kappa}{8\pi} dA + \Omega dJ \tag{2}$$

Here κ is the surface gravity of the event horizon. Ω is the angular velocity of the event horizon and J is the angular momentum of the black hole. From the above equations one draws an analogy between the area of the event horizon and the entropy of the black hole.Also it is the property of the surface gravity that it is constant on all points of the horizon.This is similar to the condition for a body being in equilibrium at a specific temperature.Thus,the surface gravity is analogous to the temperature of the black hole. Similarly the mass of the black hole is related to the internal energy. Extending these analogies, one translates the laws of Black Hole Dynamics to laws of 'Black Hole Thermodynamics'. Now let us systematically develop the formalism and notations for the black hole thermodynamical quantities. We will be using Planck units $c = G = \hbar = 8\pi k_b = 1$.In Boyer-Lindquist co-ordinates the Kerr-Metric is expressed as

$$ds^2 = -\frac{\Delta}{\Sigma}(dt^2 - a\sin^2\theta d\phi)^2 + \frac{\sin^2\theta}{\Sigma}[(r^2+a^2)d\phi - adt]^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \tag{3}$$

where $\Delta = r^2 - 2Mr + a^2 + Q^2$, $\Sigma = r^2 + a^2 cos^2\theta$, $a = J/M \geq 0$. The locations of the two horizons are given by $r_H = r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}$. Now we define the relevant thermodynamic quantities: The entropy $S = \frac{1}{4}A_\pm = \frac{1}{8}(2Mr_H - Q^2)$, Temperature $T = \frac{\pm 16\pi\sqrt{M^2-a^2-Q^2}}{A_\pm} = \beta^{-1}$, Angular velocity $\Omega = \frac{4\pi J}{MA_\pm}$, Electrostatic potential $\Phi = \frac{4\pi Q r_H}{A_\pm}$
The thermodynamic equation of state is given by the relation of the mass to the area of the black hole

$$S = (2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - M^2Q^2}) \tag{4}$$

One can observe that the entropy $S(M, J, Q)$ is a generalized homogeneous function $S(\sqrt{\lambda}M, \lambda J, \sqrt{\lambda}Q) = \lambda S(M, J, Q)$. The 3 kinds of susceptibilities corresponding to different modes of energy exchange namely thermal,electrical and mechanical. The corresponding susceptibilities are heat capacity(C_a),moment of inertia (I_b) and electric capacitance (K_c).

These are defined below:

$$C_a = -\beta^2(\frac{\partial M}{\partial \beta})_a \tag{5}$$

$$I_b = \beta(\frac{\partial J}{\partial(\beta\Omega)})_b \tag{6}$$

$$K_c = \beta(\frac{\partial Q}{\partial(\beta\Phi)})_c \tag{7}$$

Here the subscripts denote the quantities fixed by the boundary conditions. Davies [1],[2] first pointed out that the divergence of the the heat capacity of the Kerr-Newman black hole is a mark of a second order phase transition. The expression for the heat capacity of the black hole is

$$C_{J,Q} = \frac{MTS^3}{J^2 + Q^2/4 - T^2S^3} \tag{8}$$

For Schwarzschild black hole it will be $C = -M/T$, whereas for an extremal Kerr-Newman black hole, $C_{J,Q} \to 0+$ .The heat capacity will diverge in between at $Q_c = \sqrt{3}M/2$ for a Reissner-Nordstrom black hole and at $J_c = (2\sqrt{3} - 3)^{1/2}M^2$. In general the argument is that the divergence of the susceptibility is an indication of a phase transition. There have been other proposals about a phase transitions at the extremal limit of the black hole when $M^2 = a^2 + Q^2$, based on the divergence of thermal fluctuations.Interestingly the divergence of these fluctuations are related to the divergence of the C_{JQ}^{-1}. It has been shown from the theory of non-equilibrium thermodynamic fluctuations that the divergence pointed out by Davies is only related to changes in stability and not necessarily to a critical point of a phase transition,whereas the extremal limit turns out to be a point of second order transition.In the next section we shall explore the intricacy of recognizing a critical point of second order phase transition from the non-equilibrium fluctuations ,in the context of black hole thermodynamics.

3. FLUCTUATIONS,DIVERGENCES AND CRITICAL BEHAVIOR

Kaburaki [4] uses Poincare's one-parameter series of equilibria to separate out the stable and unstable states.What one does here is consider a function called the Massieu function Ψ, which contains all the information about the equilibrium states of a system,whose infinitesimal increment is given by $d\Psi = \sum_{i=1}^{n} X_i dx_i$. Here x_i are the thermodynamic variables and X_i are conjugate to x_i .i.e in an equilibrium configuration $X_i = (\partial\Psi/\partial x_i)_{x_i'}$. Now if we consider $\hat{\Psi}$ as the Massieu function analytically continued the non-equilibrium points near the equilibrium in phase space,it is shown that its second order variation

$$(\delta^2\hat{\Psi})_{x_i} = -\sum_{k=1}^{n}\left(\frac{\partial^2\Psi}{\partial x_k^2}\right)^{-1}_{x'_k}(\delta X_k)^2 \tag{9}$$

where the subscripts are the set of variables kept constant in doing the variation. One also introduces the macroscopic distribution function for the equilibrium fluctuations

$$P(\delta X_1, \delta X_2, ...\delta X_n)d\delta X_1 d\delta X_2...d\delta X_n \propto exp(\hat{\Psi} - \Psi)d\delta X_1 d\delta X_2...d\delta X_n \tag{10}$$

With these two quantities we can get the second moment of fluctuations

$$\langle \delta X_i \delta X_j \rangle = \left(\frac{\partial^2\Psi}{\partial x_i^2}\right)_{x'_i}\delta_{ij} \tag{11}$$

One can see from the above equation that the fluctuations diverge when the second derivative diverges.Therefore at the turning point where the sign of the tangent changes through infinity in a continuous part,the variance of the fluctuation also diverges. This is what happens in the case pointed out by Davies.These divergences are indeed related to the divergences of the specific heat capacities but are not relevant to any phase transitions.Specifically, speaking in the context of a black hole,this happens for a black hole in a canonical ensemble i.e in contact with some constant temperature bath of Hawking radiation.The divergence indicates an instability associated with the exchange of heat with the bath.The result of instability may be complete absorption of the radiation or the complete evaporation of the black hole into radiation.

The other possibility is the occurrence of the divergence asymptotically at the end of a curve between the thermodynamic variables and their conjugates,which is not a turning point.This happens to be associated with a critical point of a continuous phase transition.These kind of divergences occur in the micro canonical environment of the black hole,which is basically an isolated hole.Here the second moment of fluctuations are inversely proportional to the the susceptibilities. So when they vanish in the limit of black hole becoming extremal,the fluctuations diverge as in a continuous phase transition.

4. TWO-HORIZON THEOREM OR SU-CAI-YU THEOREM

Now,with all this in the hindsight,there appears a theorem due to Su-Cai-Yu [5] on the occurrence of phase transitions in black holes. Now during any process of emission of particles be it Hawking radiation,pair creation,super-radiation or the Penrose process like described before,the parameters of the black hole change.For a Kerr-Newman black hole,these changes can be approximately expressed as

$$\delta M \propto -M^{-2}$$
$$\delta J \propto -JM^{-3}$$
$$\delta Q \propto -QM^{-3}$$

Now using these to obtain the second moments of fluctuations and defining a parameter $\eta = (r_+ - r_-)/2$, we have $\langle \delta M \delta M \rangle, \langle \delta Q \delta Q \rangle, \langle \delta J \delta J \rangle, \langle \delta M \delta Q \rangle$ tend to η as $\eta \to 0$. In the same limit $\langle \delta S \delta S \rangle, \langle \delta T \delta T \rangle, \langle \delta S \delta T \rangle$ tend to η^{-1}. So some of the second moments of fluctuations diverge as $r_+ \to r_-$. This corresponds to the continuous phase transition from an extreme to non-extreme black hole.The parameter η acts as an order parameter in the phase transition. The case of $\eta = 0$ corresponds to a symmetric phase of extremal black hole,in which the two horizons have merged. The horizon temperature is zero in this case. The $\eta \neq 0$ is a less symmetric non extremal phase,which can be described thermodynamically.Only super-radiation is possible in the extremal phase whereas in the other phase there can be both super-radiance and Hawking radiation. An immediate observation one can make is that there is no possibility of a phase transition from a black hole without two horizons as the order parameter is zero.This is true irrespective of the type of the black hole.This fact can be stated and proved as a theorem:

Theorem: "If the black hole has two horizons,then r_+ and r_- are two solutions of a quadratic equation,some second moments of fluctuations concerning the temperature and the entropy of the black hole must diverge when the outer and the inner horizons become degenerate"

Proof: Let us expand the entropy and temperature to the first order of η

$$S = C_0(J,M,Q) + C_1(J,M,Q)\eta \tag{12}$$

$$T = C(J,M,Q)\eta \tag{13}$$

$$S = X_M \delta M + X_Q \delta Q + X_J \delta J \tag{14}$$

$$X_M = \frac{\partial C_0}{\partial M} + \eta \frac{\partial C_1}{\partial M} + C_1 \frac{\partial \eta}{\partial M} \tag{15}$$

Similarly for $X_{q,J}$

$$\langle \delta S \delta S \rangle = X_M^2 \langle \delta M \delta M \rangle + X_J^2 \langle \delta J \delta J \rangle + X_Q^2 \langle \delta Q \delta Q \rangle - 2X_Q X_J \langle \delta Q \delta J \rangle - 2X_M X_Q \langle \delta M \delta Q \rangle - 2X_M X_J \langle \delta M \delta J \rangle \tag{16}$$

Since r_+, r_- are the real roots of a quadratic equation we can write for a function $f(M, J, Q)$, $\eta = (r_+ - r_-)/2 = [f(J, M, Q)]^{1/2}$.Therefore,we have for $\eta \to 0$

$$\frac{\partial\eta}{\partial J} \sim \eta^{-1}, \ \frac{\partial\eta}{\partial M} \sim \eta^{-1}, \ \frac{\partial\eta}{\partial Q} \sim \eta^{-1} \tag{17}$$

Finally,we get $\langle\delta S\delta S\rangle \sim \eta^{-1}$,$\langle\delta M\delta M\rangle \sim \eta$,$\langle\delta Q\delta Q\rangle \sim \eta$,$\langle\delta J\delta J\rangle \sim \eta$. Doing the same calculations for T, we find $\langle\delta T\delta T\rangle$,$\langle\delta Q\delta Q\rangle$,$\langle\delta T\delta S\rangle$ will diverge $\eta \to 0$. Since the divergences come from the linear terms,the higher order terms in η do not affect.Also the proof is independent of the details of the emission process by which the parameters of the black hole changes. Thus in the pretext of studying the phase transitions in black holes, we have also gained clarity in the interpretation of the divergences of the the susceptibilities as phase transitions.

5. CRITICAL BEHAVIOR AND SCALING LAWS

Let us define some of the relevant response coefficients,which will be related to the susceptibilities as follows:

$$\bar{\chi}_1 \equiv \left(\frac{\partial^2 S}{\partial M^2}\right)_{JQ} = -\frac{\beta^2}{C_{JQ}} \tag{18}$$

$$\bar{\chi}_2 \equiv \left(\frac{\partial^2 S}{\partial J^2}\right)_{MQ} = -\frac{\beta}{I_{MQ}} \tag{19}$$

$$\bar{\chi}_3 \equiv \left(\frac{\partial^2 S}{\partial Q^2}\right)_{MJ} = -\frac{\beta}{K_{MJ}} \tag{20}$$

Near the critical points these quantities obey certain power laws. To define the critical exponents,let us adopt the following order parameters: $\eta_M = \beta_+ - \beta_-$, $\eta_J = (\beta\Omega)_+ - (\beta\Omega)_-$, $\eta_Q = (\beta\Phi)_+ - (\beta\Phi)_-$. However these do not go to zero at the critical point but diverge.This is because the temperature of the black hole goes to zero in this limit.Let us define the following terms for simplifying the notation: $M = M_X(1+\epsilon_M)$, $J = J_X(1-\epsilon_J)$,$Q = Q_X(1-\epsilon_Q)$ with $J_X = M\sqrt{M^2 - Q^2}$ and $Q_X = \sqrt{M^4 - J^2}/M$. ϵ's are small deviations in the parameters. Some of the relevant scaling laws:

$$\bar{\chi}_1 \sim \epsilon_M^{-\alpha} \ (J = 0/Q = 0)$$

$$\bar{\chi}_2 \sim \epsilon_M^{-\gamma} \ (Q = 0)$$

$$\eta_J \sim \epsilon_M^{\beta} \ (Q = 0)$$

$$\eta_J \sim \epsilon_J^{\delta^{-1}} \ (Q \neq 0)$$

There are other scaling laws too for which the reader is referred to the [3].The critical exponents for the Kerr-Newman black holes are

$$\alpha = 3/2,\ \beta = 1/2,\ \gamma = 3/2,\ \delta = -2 \tag{21}$$

From these we have the equalities related to the scaling laws of first kind:

$$\alpha + 2\beta + \gamma = 2,\ \beta(\delta - 1) = \gamma \tag{22}$$

6. THERMODYNAMIC GEOMETRY

We saw that using the thermodynamic fluctuation theory gave us insights into the stability and critical behavior. Now the whole formalism can be put into the language of geometry as was done by Ruppeiner [6]. We will briefly see how this is done but not get into the details of its applications. Consider a black hole and its environment or the 'universe'.The total entropy is given by $S_{tot} = S_{bh} + S_e$. We define $F_\alpha \equiv \frac{\partial S_{bh}}{\partial X^\alpha}$,where $X^\alpha = (M, J, Q)$. Now expanding the total entropy to the second order in fluctuations gives:

$$\Delta S_{tot} = F_\mu \Delta X^\mu + F_{e\mu}\Delta X_e^\mu + \frac{1}{2}\frac{\partial F_\mu}{\partial X^\nu}\Delta X^\mu \Delta X^\nu + +\frac{1}{2}\frac{\partial F_{e\mu}}{\partial X^{e\nu}}\Delta X_e^\mu \Delta X_e^\nu \tag{23}$$

The linear terms vanish due to conservation laws and for a very large environment the second quadratic term is negligible compared to the first.Therefore the above equation can now be written as

$$\frac{\Delta S_{tot}}{k_B} = -\frac{1}{2} g_{\mu\nu}(\Delta X^\mu)(\Delta X^\nu) \tag{24}$$

where the symmetric matrix g is given by

$$g_{\alpha\beta} \propto \frac{\partial^2 S}{\partial X^\alpha \partial X^\beta} \tag{25}$$

From this we can develop a formalism of Thermodynamic Riemannian geometry by defining the line element as

$$\Delta l^2 = -\frac{2\Delta S_{tot}}{k_B} = g_{\mu\nu}\Delta X^\mu \Delta X^\nu \tag{26}$$

This unitless positive definite(under stability) line element can be given a physical interpretation: Farther apart the states,less probable are the fluctuations between the states. Once a metric is defined, one can go ahead and calculate the curvature of the given geometry.The thermodynamic curvature obtained can be given various interpretation as the range of interaction, the correlation volume etc in various contexts.Refer [6]and references therein for detailed expositions.For most of the ordinary thermodynamical systems the curvature is negative. But interestingly for the Kerr-Newman black hole,it is positive.Even more curiously it is also positive for a Fermi gas.There have been attempts to give correspondence between a two dimensional Fermi gas and a Kerr Newman black hole.

7. BLACK HOLE ENTANGLEMENT ENTROPY AND RENORMALIZATION GROUP

A last note about a very recent development[7] but not with complete details. We have seen that the phase transitions at the extremal point is also related to the change in the contributions to the radiation emitted from the event horizon. In the extremal phase there is purely superradiance whereas in the non-extremal phase there are Hawking radiation and other components of emission.Very recently there has been an attempt to see the possibility of partitioning the entropy of the black into the contribution from gravitational effects alone and another contribution from the entanglement entropy of the quantum fluctuations near the horizon. The key points are as follows: There appears to be two contributions to the black hole entropy.Firstly one due to the gravitational field itself in the absence of any fields. This is the entropy given by the Beckenstein-Hawking formula.Secondly,the contribution from the quantum fields in the black hole background.This contribution to the thermal entropy is also proportional to the area,in the leading order,but with a diverging co-efficient. Apparently this contribution arises from the one loop correction to the thermal partition and can be considered as the entanglement entropy across the horizon,of the quantum fields in the global pure state. The above quoted reference tries to study the entanglement entropy within the framework of renormalisation group,in a setting where its contribution is inherently finite.They consider a matter field minimally coupled to gravitation and the key idea is to introduce a Wilsonian RG cut off scale and define an effective action for the metric,which excludes the effects of IR excitation below the scale.They try to sidestep the unknown microscopic physics of the UV cutoff and deal only with the finite quantities,by partitioning out the degrees of freedom into those with momenta greater than the intermediate scale,which is much lesser than the UV cut-off.Those degrees of freedom can be integrated and absorbed into the effective action.The paper tries to study whether such an approach can be lead to an interpretation of the BH entropy with the entanglement entropy or not. This development seem to be interesting but beyond the current scope of the author for a detailed elucidation.The interested reader can study the reference [7].

8. SUMMARY

Black Holes being characterized by just 3 parameters of mass,charge and angular momentum,are almost comparable to elementary particles having just mass,charge,spin.One the one hand, low energy systems being complexly constituted by such elementary particles,have shown amazing richness and contrast in their equilibrium thermodynamical and critical behaviour.At the same time we have seen black holes being 'simple'(!?) objects, have surprisingly show such behavior similar to these complex systems. In studying the phase transitions of black holes,we encountered some intricacies in the interpretation of critical behaviour and fluctuations. The use of non-equilibrium fluctuation theory provided some insight into these matters that the diverging susceptibilities need not always mean a critical point. Also,interesting we saw that two different phases of the Kerr-Newman black hole corresponded to the change in the contributions to the radiation from the horizon. It might be possible that further investigations into the field can lead to some insight into the hawking radiation related problems.Though it might be a hyperbole,but following the tradition of how most of the current age articles on black holes end,let us say there might be remote or close links to even microscopic theory of gravity!

BIBLIOGRAPHY

1. Davies,P C W, Proc. R. Soc. London A 353, 499 (1977).

2. Davies P C W,Rep. Prog. Phys. 41, 1313 (1978).

3. Kaburaki O , Physics Letters A,217 (1996).

4. Kaburaki O ,Gen. Rel. and Grav. 28,7,(1996).

5. Su R-K,Cai R-G,Yu P K N,Phys. Rev.D,50,4(1994)

6. Rupperiner G , Phys. Rev. D,78 ,024016 (2008).

7. Jacobson T, Satz A,Phys.Rev.D,87,084047(2013)