Non-Fixed Point Renormalization Group Flow

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Abstract: The renormalization group flow of most systems is characterized by attractive or repelling fixed points. Nevertheless, some systems can trace a different trajectory in coupling constant space corresponding to limit cycles or chaotic flow. I will focus on these types of non-conventional behaviours in the one-dimensional Ising model with complex coupling constants, and Efimov states. We map the regions of chaotic, normal, and point out the limit cycle flow spots for these types of systems.

1 Introduction

We approach the concept of non-fixed point renormalization group in the following way.

First by introducing some basic concepts of the renormalization group theory. Next, using the transfer matrix method to solve the one-dimensional Ising model. Then applying the renormalization group theory to these results in order to get the corresponding recursion relation. Next I go over the conditions for which the coupling constant flow is chaotic and what restrictions these can impose on the coupling constants (namely a complex magnetic field).

I close with the concept of Efimov states, what they are, and their relationship to non-fixed point renormalization group. Since this is a real system, experimental evidence for their existence will be presented.

2 The Renormalization Group

In 1971 K.G. Wilson developed the renormalization group theory [1]. The theory gives rise to critical exponents and explains universality. Initially conceived to show the relationship between coupling constants at different length scales, today RG is used not only in a variety of the branches of physics, but has also found applications in the fields of mathematics and biology.

Renormalization is done in two steps: coarse graining and rescaling. The former is an integration over short range degrees of freedom, shorter than l. The later is a rescaling of the distance back to the original length scale. Using the notation and approach of [2] start with the Hamiltonian:

$$H = \sum_n K_n \Theta_n\{S\} \tag{1}$$

With K_n the coupling constants, and $\Theta_n\{S\}$ the local operators, functionals of the degrees of freedom $\{S\}$, (S_iS_j in the Ising model, for example). We consider a renormalization group transform R_l which change the coupling constants as the length scale changes:

$$[K^{'}] \equiv R_l[K] \tag{2}$$

The new system will look like the one before renormalizing, but with a new effective Hamiltonian with different values of the coupling constants, and coarse grained $\{S\}$. The number of degrees of freedom N is reduced by a factor l^d. The coarse graining can be thought of explicitly as a partial trace of the degrees of

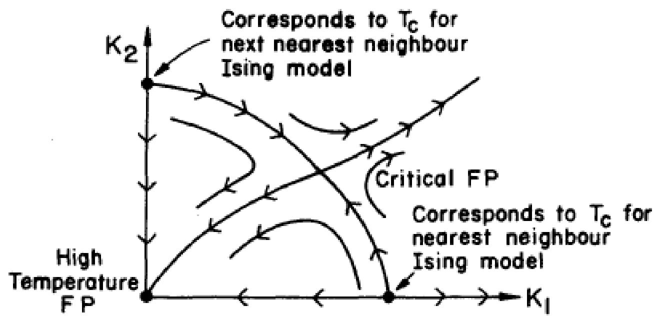

Figure 1: Taken from [2]. Coupling constant flow diagram for an Ising model with nearest neighbor (K_1) and next nearest neighbor (K_2) interactions. Fixed point correspond to different parts of the phase diagram (critical points, bulk phases, etc.)

freedom. It is imperative that we pick it such a way that we conserve the original symmetries of the Hamiltonian, and that the partition function is invariant under the renormalization. These facts will be used below.

Another concept prudent to introduce in this section is the notion of a fixed point. This is a value for our coupling constant such that:

$$[K^*] = R_l[K^*] \tag{3}$$

If we know R_l, linearizing around a fixed point will give us eigenvectors and eigenvalues for the coupling constants. Thus we can map the flow of each K_n. A positive eigenvalue of K_n gives it a flow away from the fixed point, for a negative eigenvalue, the fixed point is an attractor. Figure 1 shows the flow for an Ising model with nearest neighbor (K_1) and next nearest neighbor (K_2) interactions.

Though in general, deriving the particular recursion relations may be an extremely difficult task, the basic concept behind the renormalization group seems simple. Take a picture and frame it, put it on a wall. Stand away from the wall and take a picture of the picture. Develop it and amplify it so that the image fits the original frame. Cut it out, frame the cut out and repeat the process. Eventually, intuitively, the image becomes so blurry that one cannot distinguish between two consecutive steps. We have reached a fixed point. (And, like in formal renormalization, we cannot recover information lost by integrating out the degrees of freedom after each step.)

What if the outcome of doing this an arbitrarily large number of times wasn't a blur? What if after a performing the process a number of times the image re-surfaced– that is –the coupling constants returned to their starting values? Or what if, like an infinitely complex kaleidoscope, the coupling constants kept changing with no sign of a fixed point? Limit cycles and chaotic renormalization group flow in coupling constant space, respectively, are descriptions of these cases. Although Wilson didn't treat them in his 1971 paper, he was aware of and mentioned these possibilities.

3 Chaos in the Ising Model

One of the simplest examples of renormalization group flow that doesn't necessarily lead to fixed point trajectories is realized in the one-dimensional Ising model with periodic boundaries. A major pedagogical advantage of this model is the fact that the renormalization group recursion relations can be solved analytically (whereas this is not the case for most systems). By staring at a closed form solution, we develop a physical intuition for the behaviour of the system. In the following subsections I'll work out the model to get the recursions using the transfer matrix method, then introduce chaotic behavior by allowing complex coupling constants.

3.1 1d Ising Using the Transfer Matrix Formalism

With computation of thermodynamic properties of the system in mind, we introduce the transfer matrix method. For a more thorough discussion see, for example, Chapter 3 of [2] (which I follow below). The trick lies in the factorization of the partition function.
The nearest neighbor Ising Hamiltonian for a chain of sites is:

$$-H_\Omega = H\sum_i S_i + J\sum_{<ij>} S_iS_j \tag{4}$$

The partition function is:

$$Z_N(h,K) = \mathrm{Tr} e^{h\sum_i S_i + K\sum_i S_iS_{i+1}} \tag{5}$$

Where and $h \equiv \beta H$ and$K \equiv \beta J$. And since we are using periodic boundaries, $S_{N+1} = S_1$.
Factorizing the partition function as

$$Z_N(h,K) = \sum_{S_1}\dots\sum_{S_N}[e^{\frac{h}{2}(S_1+S_2)+KS_1S_2}]\cdot[e^{\frac{h}{2}(S_2+S_3)+KS_2S_3}]\cdot\ldots\cdot[e^{\frac{h}{2}(S_N+S_1)+KS_NS_1}] \tag{6}$$

Each of the terms will be the elements of our transfer matrix $\mathbf{T}$:

$$T_{S_nS_{n+1}}e^{\frac{h}{2}(S_n+S_{n+1})+KS_nS_{n+1}} \tag{7}$$

Since the individual spins have a value of ± 1 we have only four unique types of terms, and we can write our transfer matrix as:

$$\mathbf{T} = \begin{pmatrix} T_{11} & T_{1-1} \\ T_{-11} & T_{-1-1} \end{pmatrix} = \begin{pmatrix} e^{h+K} & e^{-K} \\ e^{-K} & e^{-h+K} \end{pmatrix} \tag{8}$$

We then see that the partition function (3) is the trace of the $\mathbf{T}$ matrix

$$Z_N(h,K) = \sum_{S_1} \dots \sum_{S_N} T_{S_1 S_2} T_{S_2 S_3} \dots T_{S_N S_1} = \sum_{S_1} T^N_{S_1 S_1} = \mathrm{Tr}(\mathbf{T}^N) \tag{9}$$

We can diagonalize $\mathbf{T}$ using a matrix $\mathbf{S}$ by

$$\mathbf{T}' = \mathbf{S^{-1}TS} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{10}$$

λ_1 and λ_2 are the eigenvalues of $\mathbf{T}$. Now, the properties of trace state that

$$\mathrm{Tr}(\mathbf{T}^N) = \mathrm{Tr}(\mathbf{T}'^N) \tag{11}$$

Therefore

$$Z_N(h,K) = \mathrm{Tr}(\mathbf{T}^N) = {\lambda_1}^N \left(1 + \left(\frac{\lambda_2}{\lambda_1}\right)^N\right) \tag{12}$$

The eigenvalues are easily computed to give

$$\lambda_{1,2} = e^K \left[\cosh h \pm \sqrt{\sinh^2 h + e^{-4K}}\right] \tag{13}$$

We see that if they are non-degenerate then, one being bigger than the other, in the thermodynamic limit:

$$\lim_{N\to\infty} Z_N(h,K) \simeq \lambda_1^N \tag{14}$$

We have arrived at a useful form of the partition function! From this we can readily derive the free energy, magnetization, or other thermodynamic quantities we might be interested in.

3.2 Decimation

Now we do a little renormalization. Decimation is a process in real space, we obtain the recursion relation coarse graining the lattice. *Exercise 9-3* in [2] integrates the short distance degrees of freedom by taking the trace over only even numbered spins in the partition function above. The number of sites has been reduced by a factor of 2. Keeping the partition function invariant, as it should be, we write:

$$Z_{N/2}(h' K') = A^N Z_N(h,K) \tag{15}$$

With A^N a normalization constant (a shift in the zeroth of energy) which I will drop. We can write the equation above in matrix form:

$$\begin{pmatrix} e^{h'+K'} & e^{-K'} \\ e^{-K'} & e^{-h'+K'} \end{pmatrix} = \begin{pmatrix} e^{h+K} & e^{-K} \\ e^{-K} & e^{-h+K} \end{pmatrix}^2 \tag{16}$$

This gives our recursion relations for the two coupling constants:

$$e^{2h'} = e^{2h}\frac{\cosh(2K+h)}{\cosh(2K+h)} \tag{17}$$

$$e^{4K'} = \frac{\cosh(4K) + \cosh(2h)}{2\cosh^2(h)} \tag{18}$$

We could now get the eigenvalues of the coupling constants and map the flow diagram towards the fixed points of our coupling constants. Instead we pause. Instead we look for the chaotic conditions of our system.

3.3 Chaos from the Complex Magnetic Field

In this section we introduce chaotic behavior by letting the magnetic field be complex. Showing this requires a bit of mathematical intuition, one can go through the next computation, then look back at the equations to confirm chaotic behaviour. I follow the steps on [3]. Starting with a renormalization invariant

$$m \equiv 1 + e^{4K}\sinh^2 h = m' \tag{19}$$

the recursion relation of the previous subsection reads:

$$e^{4K'} - 1 = \frac{1}{4}\frac{(e^{4K}-1)^2}{[(e^{4K}-1)+m]} \tag{20}$$

Defining

$$x = -\frac{m}{(e^{4K}-1)} \tag{21}$$

For positive m and K, $-\infty < x < 0$, and the recursion relation

$$x' = 4x(1-x) \tag{22}$$

is obtained. This recursion relation is chaotic for $0 < x < 1$. This can be seen if we chose $x = \sin^2(\pi\psi)$, with $0 < \psi < \frac{1}{2}$. Plugging this into the recursion for x and using trigonometry we find

$$\sin(\pi\psi') = \sin(2\pi\psi) \tag{23}$$

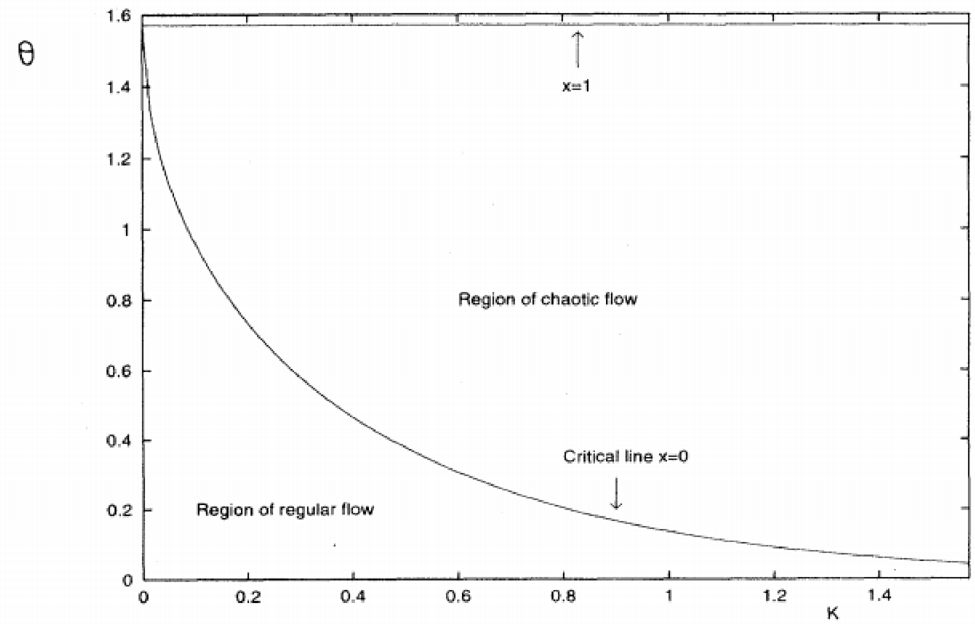

Figure 2: From [3]. Phase diagram describing the regions of chaotic flow. θ is the magnitude of the pure imaginary magnetic field h, K is our nearest neighbour ferromagnetic coupling. Note that limit cycles exist in the region of chaotic flow for integer values of ψ. The region of chaotic flow increases as $T \to 0$.

For an initial rational value of ψ, this leads to periodic orbits. For an initial irrational value of ψ the trajectory never repeats, ψ is chaotic. Since the cardinality of the irrationals is greater than that of the rationals, ψ is chaotic for almost all initial values.

Now that we know the chaotic conditions for these parameters, we backtrack and figure out what they mean in terms of our coupling constants. For chaos we need $0 < x < 1$ which $m < 0$. If K is positive and real (1d Ising ferromagnet) then real values of h will never give $m = 1 + e^{4K}\sinh^2 h$. On the other hand, if h is pure imaginary $i\theta$, then $\sinh^2 h \to \sin^2\theta$. Then $0 < x < 1$ and $m < 0$ for:

$$\sin^2\theta > e^{-4K} \tag{24}$$

where θ is the magnitude of our pure imaginary h. This is the condition for chaos plotted in Figure 2.

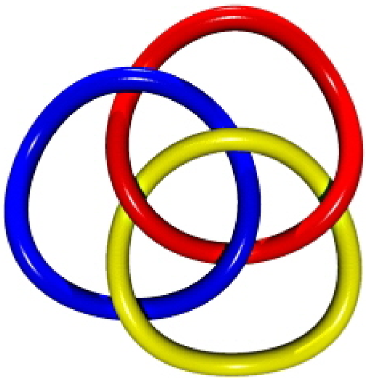

Figure 3: From [5]. The Efimov three-body bound state in a two-body potential resembles a Borromean ring. Breaking one ring destroys all the bonds.

3.4 Remarks

I chose to go through this calculation in detail because the model is extremely simple and can be solved analytically. The reader can develop physical intuition on each step, which is great as a first exposure, or just to review basic knowledge. Nevertheless, the introduction of a complex magnetic field begs the question of its meaning. In [3], Dolan references examples where letting a parameter be imaginary may lead to deeper knowledge of a theory. (For instance, imaginary electric charge leads to a negative fine structure constant [4].) He also points out that letting coupling constants be complex is essential to solve some two-dimensional statistical models. Additionally, after the calculation above, he goes on to discuss the consequences of having a complex K. In the end, though I understand the approach, I am not comfortable with the concept of an imaginary field (perhaps we require it because the model is too simple). Until I understand, I'll be whether the boundary between physics and manipulation of equations was crossed (is Newton's second law meaningful for complex mass?).

4 Efimov States

A three-body Efimov state occurs on a two-body potential too weak to hold a two-body bound state (Figure 3). Efimov showed [6] that if tuned to resonance such that the scattering length a is large compared to the range l of the potential then there is a number of three-body states available spaced geometrically over the range $\hbar^2/ml^2$ and $\hbar^2/ma^2$. In fact, the potential can hold an infinite amount of three-body bound states as $a \to \pm\infty$, equivalently as we get arbitrarily close to the zeroth of energy at the continuum . He showed that the ratio of the energy of

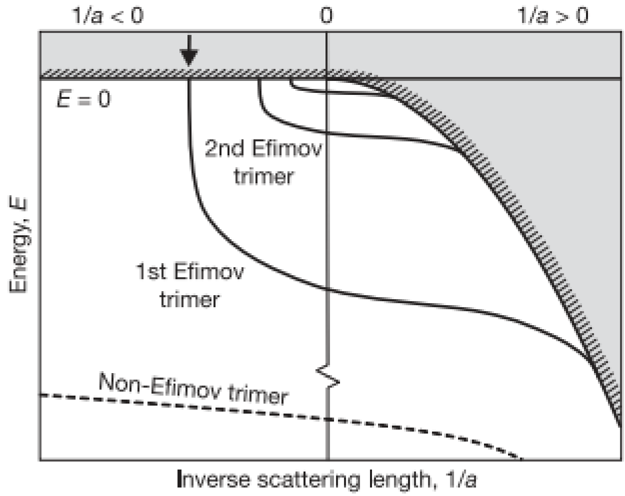

Figure 4: From [8]. Efimov three-body states appearance as a function of inverse scattering length and energy. The shaded region denotes the continuum limit for trimers ($1/a < 0$) and for an atom and a dimer ($1/a > 0$). As we get closer to resonance more states appear.

two consecutive bound states is a universal number, $e^{2\pi s_o} \simeq 515$.

That last statement implies Efimov states can be understood in terms of limit cycles. That is, as we vary the scattering length, moving closer to resonance we get a *discrete* scale transformation: $x \to (S_0)^n x$ with integer n, and where S_0 is a discrete scaling factor. In principle, we are able to accept an arbitrary, but (obviously) discrete, number of three-body bound states in our potential as we arbitrarily approach zero in the inverse scattering length $1/a$. We can physically alter the length scale of the system (as opposed to a coarse graining and rescaling that change the couplings but must not change the observables). A pictorial image of this is presented in Figure 4. For a more complete discussion of the characteristics on the limit cycles Hammer gives a discussion in [7].

Though elusive for many years, the Efimov state has been confirmed relatively recently using ultracold caesium atoms [8], or even more recently in ultracold ^6Li atoms [9]. The Cs atoms in [8] were placed in the potential using a crossed optical dipole trap. Fine tuning the atomic interactions was achieved using Feshbach resonances by changing the magnetic field. Evidence for Efimov states was demonstrated by observing resonances in three-body state recombination (Figure 5). Kraemer measured recombination atomic losses in a gas of trapped atoms when crossing the trimer to continuous boundary in Figure 4.

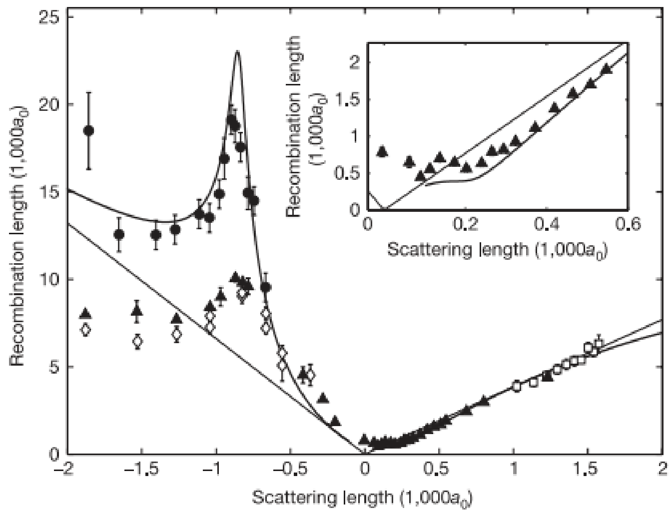

Figure 5: From [8]. Triatomic Efimov resonance. Recombination length for a trimer versus scattering length.

5 Conclusion

The concept of non-fixed point renormalization group flow was introduced using two approaches. First, in a pedagogical way, the conditions for chaotic behaviour in the one-dimensional Ising model were derived. Then, striving for meaning, the implications on the renormalization group for an Efimov state were discussed.

With the first approach we attempt to demystify a theory leading to the non-fixed point flow, but not its consequences. Since fixed points correspond to bulk phases, critical points, triple points, or other locations in our phase diagram, coupling constants not renormalizing to them seem strange. One might think that not much can be learned about the system for non-conventional flow, after all the renormalization is useful to determine the asymptotics of systems. Hence we present the Efimov effect, in which the renormalization group provides a way to understand it's physics. The fact that the Efimov effect is a limit cycle is a rare example of real RG limit cycles.

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