# Phases of the Bilinear-biquadratic Spin-1 Chains

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**Abstract**

Measurements done on LiVGe2O6 by Millet et al. [Phys. Rev. Lett. 83, 4176 (1999)] on phases of the Singlet ground state of the Spin-1 Chain can be described by the Bilinear-biquadratic spin-1 chain. I will discuss these results, among the results of others and talk about theoretical calculations done by A. Lauchli [Phys. Rev. B 74, 144426 (2006)] that describe the phase diagram of the bilinear-biquadratic model. At a particular point in the phase diagram the model becomes SU(3) globally symmetric and is known as the Lai-Sutherland model, it is Bethe ansatz solvable as shown in [A Schmitt et al., J. Phys. A: Math. Gen. 29 (1996) 3951-3962]. Finally I will describe the thermodynamics of the model as where calculated by Lou et al. [Phys. Rev. Lett. 85, 11 (2000)].

# 1 Introduction

In the following literature review, I will go over a particular model for Spin-1 chains, namely, the bilinear-biquadratic spin-1 chain. This model usually appears as an effective description of interacting systems such as [1] and [2] and more interestingly as a description of actual physical situations such as $^{23}$Na atoms in optical lattices [5]. I will explain how the model is confirmed experimentally for the first two cases.

The phase diagram for the bilinear-biquadratic chain, is "very rich", it is studied in detail using numerical methods by A. Lauchli et al. in [4] building on Renormalization Group calculations performed by C. Itoi et al [6]. At a particular point of the diagram, the model exhibits a global SU(3) symmetry, which is Bethe anzats solvable. I will devote a subsection to this.

# 2 Experimental Realizations (some examples)

## 2.1 Some experiments

A key feature that exists on Spin-1 chains that is not realized on spin-1/2 chains, is the existence of a Haldane gap[1]. This provides natural experimental testability for the realization of such a system. Katsumata[1] based on the idea that, since the ground state and the first excited states are a singlet and a triplet respectively, proposed that as the energy of a triplet state is lowered, due to the magnetic field being lowered, it will match the energy of the ground state at a critical field $H_c$. Experimentally this is done by measuring the magnetization, which will be zero for $H < H_c$ and will turn on at $H_c$.

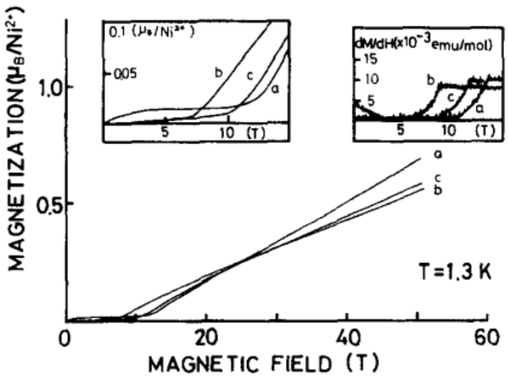

Figure 1: M-H curves at T=1.3K for $Ni(C_2H_8N_2)_2NO_2(ClO_4)$. (Image found in [1])

The figure to the right, also found in [1], shows the M-H curve for $Ni(C_2H_8N_2)_2NO_2(ClO_4)$, which is a S=1 1D Heisenberg antiferromagnet. The data matches the description of the provided data. The appropriate

Hamiltonian for this system is

$$H = -J\sum_{i=1}^{n} S_i \cdot S_{i+1} + D(S_i^z)^2$$

where J is the exchange interaction, N the total number of spins and $D > 0$ the single ion anisotropy constant.

In order to understand the ground state of similar S=1 systems, Affleck et al. [2] showed that the Valence Bond Solid(VBS) state is the ground state of the Hamiltonian

$$H = -J\sum_{i=1}^{n} S_i \cdot S_{i+1} + (S_i \cdot S_{i+1})^2/3$$

The spin singlet state of the S=1 chain can be written with two valence bonds emanating from each site[1]. The energy of both states is also similar. Based on this, the VBS model would be a good approximation for the S=1 Anti-ferromagnet. This also was also tested experimentally by [1] with satisfactory results.

## 2.2 $LiVGe_2O_6$

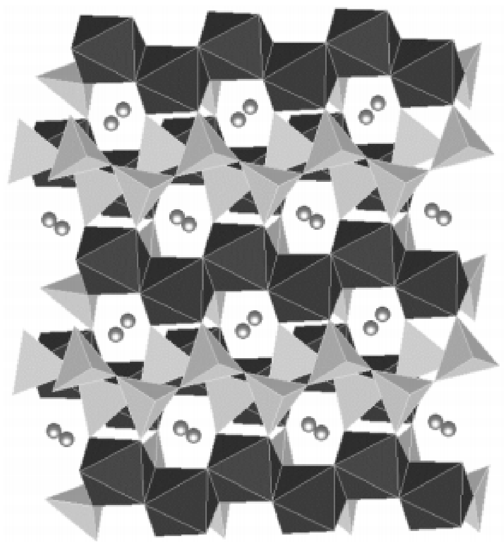

Figure 2: Polyhedra representation of $LiVGe_2O_6$ (figure from [3])

The most general Hamiltonian describing an isotropic Spin-1 chain is

$$H = -J'\sum_{i=1}^{n} S_i \cdot S_{i+1} + J''(S_i \cdot S_{i+1})^2$$

whose phase diagram can be studied by performing the reparametrization $J' = J$, $J'' = -\beta J$. For this parametrization, the Haldane gapped phase occurs for $J > 0$ and $-1 < \beta < 1$ while the dimerized phase happens for $\beta > 1$ for $J > 0$ and $\beta < 1$ for $J < 0$ [3]. I will return to the phase diagram in the next section.

Most bilinear biquadratic systems, display a small $\beta$ which limits the experimental studies of the full phase diagram[3]. P. Millet, et. al. studied the Vanadium Oxide $LiVGe_2O_6$ [3] which is a spin-1 chain and has a different susceptibility than other spin chains. They suggest that this system may have a significant biquadratic exchange (large $\beta$).

The (really interesting) crystal structure (left), as explained in the article [3], consists of isolated chains of edge sharing $VO_6$ linked together by chains of corner sharing $GeO_4$ tetrahedra. The structure was also determined experimentally by [3]. Chains are connected to their neighbors through only two tetrahedra making the coupling perpendicular to the chains small. This is why the model for a 1D chain is valid.

They measured a drop in the Susceptibility at 22K which they believe is a spin-Peierls transition. The regime above this temperature is characteristic of spin-1/2 systems and not of spin-1 systems. This because spin-1/2 systems are gapless in this region, suggesting gapless modes or absence of a Haldane gap. Furthermore, when they compared their data to previous calculations, they also concluded that the system had to be a spin-1 system.This implies that either the system has other possible underlying interactions that close the gap, or the biquadratic interaction is strong ($|\beta| > 1$). The authors conclude the later based one the argument the bond interactions with other chains are weak in the regime that was studied.

This experimental realization is just another motivation to study the whole phase diagram for the bilinear-biquadratic spin-1 model.

# 3 Phase Diagram

The bilinear-biquadratic S=1 1D chain, has a "very rich" phase diagram [4]. The Hamiltonian is

$$H = \sum_{i=1}^{n} \left( cos\theta S_i \cdot S_{i+1} + sin\theta (S_i \cdot S_{i+1})^2 \right)$$

When $\theta = \pi/4$, the model acquires a global SU(3) symmetry and its called the Lai-Sutherland model. I will talk about it in a subsection below.

### 3.1 Phases of the S=1 bilinear-biquadratic chain

Experiments performed in optical lattices with $^{23}$Na by [5] as mentioned in [4], are realizations of a tunable S=1 bilinear biquadratic Chain. A. Lauchli et al. in [4] performed numerical studies of the phase diagram of the model, which I will go over.

According to [4], the well established phases of the phase diagram, (right), are a Haldane, a ferromagnetic one, and the dimerized one. As they theorize (and explain), the $\pi/4 \leq \theta \leq \pi/2$ phase is gapless with spin quadrupolar correlations. The other phase near $-3\pi/4$, they explain, is possibly a spin nematic phase.

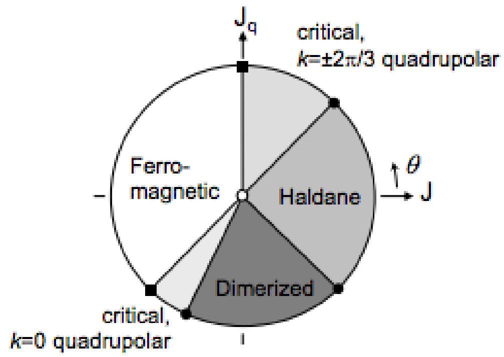


Figure 3: Phase diagram of the bilinear-biquadratic lattice (Image from [4])

At the $\theta = \pi/4$ point, as mentioned before, the model has an global SU(3) symmetry. C. Itoi et al., studied the behavior of the model around this point in the following manner [6].
Non-abelian Bosonization of the low energy behavior around the $\theta = \pi/4$ point maps the model to a gapless SU(3) Wess-Zumino-Witten (WZW) theory. Considering deviations from this point as perturbations they proceeded to do Renormalization Group calculations to find the behavior around the point.

The action they use is a perturbed Conformal Field theory,

$$A = A_{SU(\nu)} + \sum_{i=1}^{2} g_i \int \frac{d^2 z}{2\pi} \Phi^{(i)}(z, \bar{z})$$

where the first term is the SU(3) symmetric WZW action and the second part,

$$\Phi^{(1)}(z, \bar{z}) = \frac{2}{\sqrt{\nu^2 - 1}} J_L^A(z) J_R^A(\bar{z})$$

and

$$\Phi^{(1)}(z, \bar{z}) = \frac{4T^A_{\alpha\beta} T^B_{\alpha\beta}}{\sqrt{\nu^2 - 1}} J_L^A(z) J_R^A(\bar{z})$$

These perturbations are deviations from the SU(3) symmetric model. The T's are the generators for the Spin-$\nu$ algebra and the J's are the fermion currents used for the bosonization procedure (The spin Hamiltonian is written in terms of fermions and then bosonized).

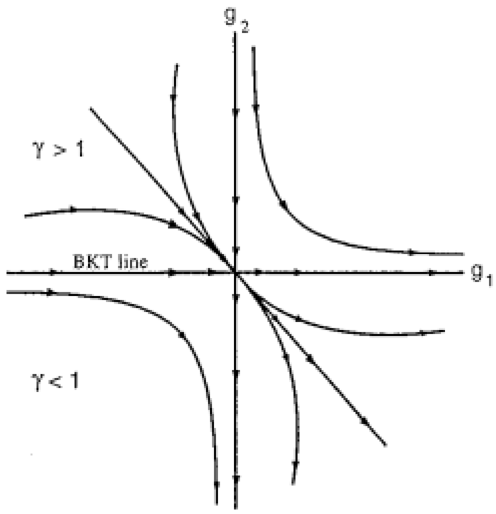


Figure 4: Renormalization Group Flows (Image from [6])

The Renormalization Group Flows for $\nu > 2$ (left), as calculated by Itoi et al. [6]. In the diagram, $\gamma$ corresponds to $Tan\ \theta$ (the $\theta$ from the bilinear biquadratic chain). $g_2$ is proportional to $\gamma - 1$. As can be seen from the flow diagram, there is a fixed point at $g_1^* = g_2^* = 0$.

They calculate three different regimes: (I quote) (i) $g_2 = 0$; SU($\nu$) symmetric and asymptotically nonfree, (i) $g_2 > 0$; SU($\nu$) asymmetric and asymptotically nonfree, (ii) $g_2 < 0$; SU($\nu$) asymmetric and asymptotically free. The corresponding phases around $\theta = \pi/4$ are then, massive for $\theta > \pi/4$ and massless for $\theta \leq \pi/4$.

As explained before, Lauchli et al. built on this calculations to construct the diagram above using different numerical methods [4].

First, they calculated quadrupole (ie. $(S^z)^2(-k)(S^z)^2(k)$) expectation values and compared them to the spin ($S_z(-k)S_z(k)$) correlations at a particular k. Their results, shown right, show that the quadrupolar interaction becomes dominant for $\pi/4 < \theta < \pi/2$. This is why they call this phase spin nematic. This maps with improvement the phase diagram around the SU(3) symmetric point mapped by Itoi et al.

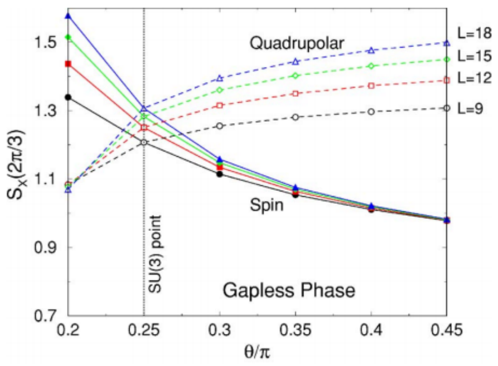


Figure 5: Quadrupolar expectation values compared to Spin ones (Image from [4])

## 3.2 The Lai-Sutherland model

The bilinear biquadratic Spin-1 model has a global SU(3) symmetry when $\theta = \pi/4$. This model is the so called Lai-Sutherland model.

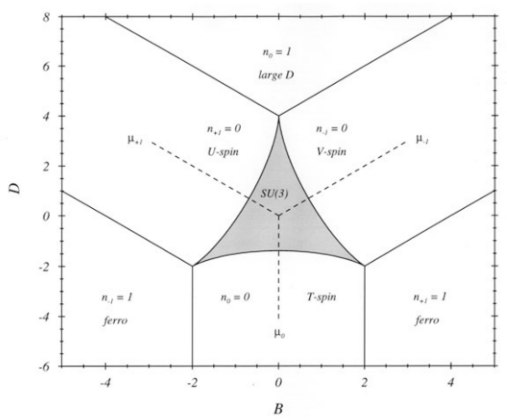


Figure 6: Phase diagram of SU(3) symmetric bilinear biquadratic spin-1 chain model (Image from [7])

This model can be thought of as a generalized Heisenberg model to S=1[7]. Upon the inclusion of external fields, Schmitt et al. studied its phase diagram[7] (left). The model they studied is

$$H = H_0 + BS_z + DS_3^{(2)}$$

where

$$S_z = \sum_i S_{zi}$$

and

$$S_z^{(2)} = \sum_i (S_{zi})^2$$

This model, as they mention, can be solved analytically via a bethe ansatz. This is what allowed Itoi et al. to map to a WZW theory[6].

In the phase diagram, according to Schmitt et al., there are essentially 3 different regions: Large D, ferromagnetic regime and the regime that corresponds to the 3 SU(2) subalgebras of SU(3). The latter are labeled U,V,T as they are called in particle physics.

# 4 Conclusions

The bilinear-biquadratic lattice model describes a wide variety of physical systems. Some of those, such as $^{23}$Na in an optical lattice, can actually match the model together with its tunable parameters making the search for a better understanding of the model very desirable. Other models however can be explained by particular phases of this model, still justifying the time invested in its study.

Theoretical studies were reviewed, in particular, exploring the phase diagram of the bilinear biquadratic S=1 chain and a particular state of it, namely the SU(3) globally symmetric Lai-Sutherland model. These studies where done using both numerical and exact calculations. Itoi et al.[6] made an interesting mapping to a WZW theory for the SU(3) symmetric model which they then perturbed to get a better understanding of the phases around the simetric point.

# References


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