PHYS 563 term Paper
The Flocking Transition : A Review of The Vicsek Model

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Abstract

This essay reviews important results from studies on the Vicsek model, which describes a flocking phase transition in self-propelled particles as a function of particle density or noise. The nature of the transition to collective motion is investigated, and the phase diagram in noise-density space is analyzed. It is shown that the flocking transition, which was originally considered to be continuous by Vicsek and collaborators, is actually discontinuous and fluctuation driven. The likeness of the flocking transition in the Vicsek model to a liquid-gas phase transition is discussed, and a qualitative study of the phase coexistence regime is performed. Lastly, the Vicsek model is compared to an active Ising model which shows a similar flocking transition, albeit with a different phase coexistence profile. The reason for the difference in the phase coexistence profiles of the two models is discussed and shown to be stochasticity driven.

1 Introduction

Active matter consists of self-driven particles which convert stored energy into directed motion, thus keeping the system perpetually driven out of equilibrium. Interactions among such self-propelled particles give rise to novel collective behaviour (flocking, swarming etc.), which has no equilibrium analogue. Active matter systems have consequently been the subject of great interest (and controversy) among researchers. Examples of active matter systems include bacterial suspensions, motor protein-cytoskeletal filament assays, terrestrial, aerial and aquatic flocks, vibrated granular rods etc.

Active matter consists of two kinds of self-propelled particles: polar and apolar [4]. Polar self-propelled particles have distinct heads and tails, and are propelled head-first. Apolar self-propelled particles have a head-tail symmetry, and move head-first or tail-first with equal probability. Predominantly two classes of collective behaviour can be emergent in active matter systems. The first class is polar ordering, which is observed in systems consisting of polar self-propelled particles subject to a polar alignment interaction, which aligns heads to heads and tails to tails. The ordered phase of such a system shows particles aligned along their long axis and moving collectively in a single direction (flocking). The second class is nematic ordering, which can be observed for both polar and apolar self-propelled particles, if the only interaction they are subject to is apolar in nature. Such interactions (e.g. volume exclusion) do not distinguish between heads and tails, and the resulting ordered phase has the same head-tail symmetry, even though the particles are aligned along their long axis. A cartoon of these different cases is shown in Figure 1.

In this essay we will review the simplest individual based model that describes flocking in self-propelled particles - the Vicsek model, proposed by Vicsek and collaborators in 1995 [10]. This model gained popularity because of its minimal nature and computational tractability. The Vicsek model exhibits a phase transition from an isotropic phase to a polar ordered phase at high density of particles or low noise. Despite its simplicity, the onset of collective motion in the Vicsek model is not well understood even after over two decades since it was proposed. For almost a decade, the transition to polar order in the Vicsek Model was thought to be continuous. It was later shown [2] that the transition is actually 1st order, with an intermediate coexistence regime of high density ordered bands traveling in a low density disordered background. Moreover, it was also shown recently [7] that because of this intermediate inhomogeneous regime of phase coexistence, the flocking transition in the Vicsek model is best understood as a liquid-gas phase transition. This essay is organized as follows - In section 2 we will discuss Vicsek's original paper, and summarize their findings. In section 3 we will discuss the work of Chaté and collaborators that proved that the phase transition in the Vicsek model is actually discontinuous and fluctuation driven. In section 4 we will talk about how the flocking transition is best understood in terms of a liquid-gas transition, and compare the Vicsek model to an active Ising model [8], which also exhibits similar flocking behaviour. Finally in section 5 we will close with a summary of what was discussed and a few concluding remarks.

2 The Vicsek Model

The Vicsek model (VM), propounded by Vicsek and collaborators in 1995 [10] is a seminal agent based model for the flocking transition. It consists of polar point particles in 2d which are self propelled at a constant velocity, and can move in any direction. At each time step, a particle reorients itself along the average direction of propulsion of particles in a unit radius around it, with some external noise preventing perfect alignment. The model exhibits a kinetic phase transition from a state of no net transport at high noise and low packing fractions, to a state with polar transport for low noise and

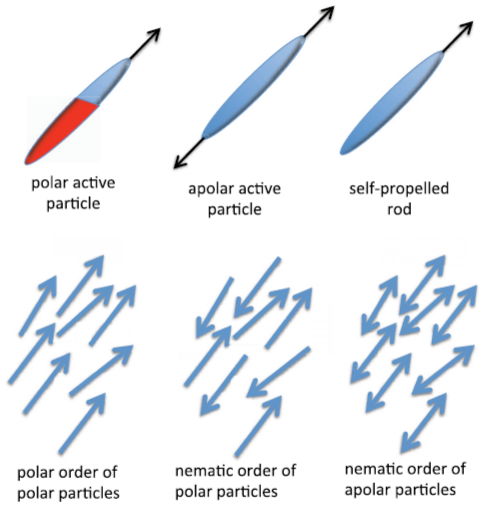

Figure 1: Cartoon picture of various types of self-propelled particles and the collective behaviour they exhibit. [Picture taken from *Marchetti et al.* Rev. Mod. Phys. 85, 1143(2013)]

high packing fraction, by the spontaneous breaking of continuous rotational symmetry. According to the classification of active matter in the previous section, the VM thus describes polar particles with polar alignment interactions, and the ordered state is polar.

The flocking transition described by the VM is analogous to the ferromagnetic transition in the Ising model. The strength of the alignment interaction (proportional to density) is like the external magnetic field, whereas the the random noise associated with the alignment is equivalent to temperature. There is an important distinction between the two however - while the Ising model is static, the VM is inherently dynamic in nature. As such the alignment of magnetic spins in the Ising model is replaced by the alignment of self-propulsion directions at every time step in the VM. This analogy between magnetic and flocking systems will be discussed further in a later section, when we talk about an *active Ising model* [8] which also exhibits dynamic flocking behaviour.

2.1 The Model

The Vicsek model consists of N particles on a 2d square lattice of linear size L. The j^{th} particle moves with a constant speed v_0, in a direction making an angle $\theta_j(t)$ with the x-axis at time t:

$$\mathbf{v}_j(t) = v_0 e^{\iota\theta_j(t)}. \tag{1}$$

At every time step, the j^{th} particle assumes the average direction of the particles in a neighbourhood defined by the interaction radius r, with some error in perfect alignment characterized by a random noise $\Delta\theta_j$:

$$\theta_j(t+1) = \langle\theta_j(t)\rangle_r + \Delta\theta_j, \tag{2}$$

where $\Delta\theta_j$ is a random number picked from the interval $[-\eta/2, \eta/2]$, and r is set to unity. The position of the j^{th} particle gets updated as:

$$\mathbf{x}_j(t+1) = \mathbf{x}_j(t) + \mathbf{v}_j(t)\Delta t. \tag{3}$$

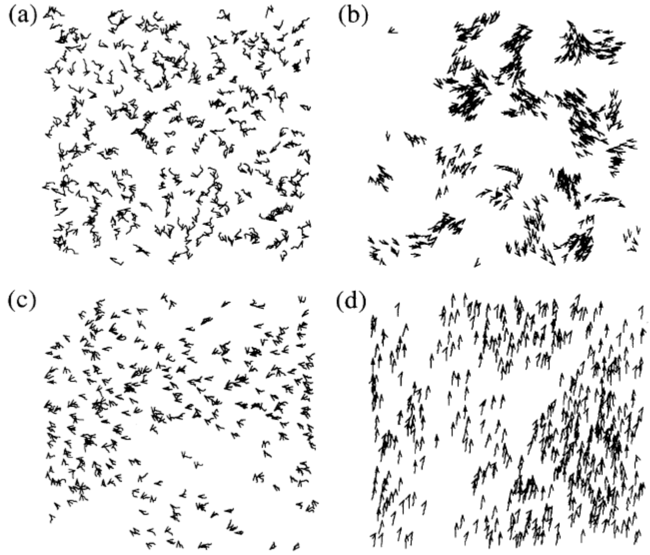

Figure 2: Velocity field for $N = 300$ self-propelled particles. (a) $t = 0, \rho = 6.12, \eta = 2.0$: initial condition, isotropic phase. (b) $t \neq 0, \rho = 0.48, \eta = 0.1$: low density-low noise. (c) $t \neq 0, \rho = 6.12, \eta = 2.0$: high density-high noise. (d) $t \neq 0, \rho = 12, \eta = 0.1$: high density-low noise limit, polar ordered phase. [Picture taken from *Vicsek et al.* Phys. Rev. Lett. 75, 1226 (1995)]

The initial condition chosen is a random distribution of N particles with random directions of motion. Thus, there are three free parameters in this model - the density $\rho = N/L^2$, the self propulsion speed v_0 and the strength of the noise η. The phase transition from isotopic to polar ordered state can be brought about by tuning either the noise or density. For less noise and high density the system is ordered, while in the other limit, the system is isotropic. The self-propulsion speed can be varied too: in the limit $v_0 \to \infty$, the system is completely mixed in a very short number of steps and we get the analogue of a mean field ferromagnet. For $v_0 \to 0$, the particles are fixed at their lattice sites, and the situation is analogous to the static XY model of spins. The flocking transition can be observed however for a range of intermediate velocities.

2.2 Vicsek's Results

Figure 2 is a snapshot of velocity field configurations for different values of noise and density, taken from Vicsek's paper [10]. The individual velocities are depicted by the small arrow, and the trajectory over the last twenty time steps by a short continuous curve. Figures 2(a) and 2(c) show the system at two different times for the same density ($\rho = 6.12$) and noise strength ($\eta = 2.0$), both of which are moderately high. Figure 2(a) shows the system at time $t = 0$, when the positions of the particles and their velocities are randomly distributed, as expected from the initial conditions imposed. The system thus starts off in the isotropic phase. Figure 2(c) shows the same system at a later time, and we observe that although the particles still move randomly because of the large noise strength, there is some amount of correlation in their movement due to the relatively high density. Figure 2(b) shows that at low densities and low noise ($\rho = 0.48, \eta = 0.1$) the particles get grouped into clusters which move in random directions. Figure 2(d) depicts the most dramatic collective behaviour, observed at high density ($\rho = 12$) and low noise ($\eta = 0.1$). In this regime, all particles are aligned and move in a single spontaneously chosen direction - this is the flocking phase which is polar ordered. Thus we see that a "flocking transition" occurs going from the limit of low density and high noise to the limit of

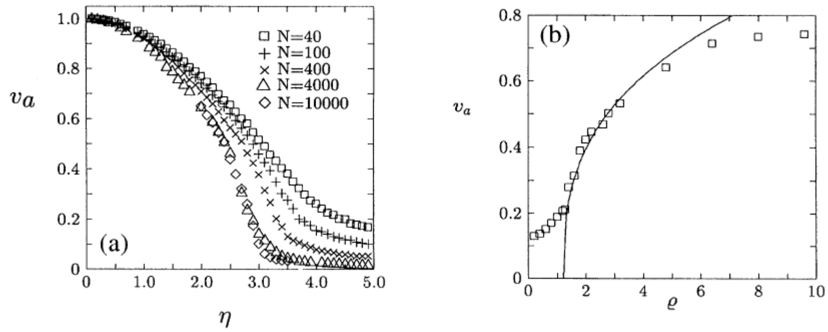

Figure 3: (a) The order parameter v_a vs noise η at fixed density, $\rho = 4$. The symbols correspond to $\Box : (N = 40, L = 3.1), + : (N = 100, L = 5), \times : (N = 400, L = 10), \Delta : (N = 4000, L = 31.6), \Diamond : (N = 10000, L = 50)$. (b) The order parameter v_a vs density ρ at fixed noise (L=20). [Picture taken from *Vicsek et al.* Phys. Rev. Lett. 75, 1226 (1995)]

high density and low noise, by the spontaneous breaking of continuous rotational symmetry.

The phase transition can be characterized by defining the average velocity to be the order parameter:

$$v_a = \frac{1}{N v_0} |\sum_i \mathbf{v}_i|. \tag{4}$$

In the isotropic phase velocities are randomly oriented, and $v_a = 0$. In the ordered phase, particles all move in the same direction and $v_a = 1$. Figure 3(a) shows v_a as a function of noise strength for a fixed density ($\rho = 4$). The density is kept fixed by varying the number of particles N and the system size L simultaneously so as to keep $\rho = N/L^2$ constant. It is clear that when noise is reduced below a certain critical value η_c for the given density, there is a transition from an isotropic ($v_a = 0$) to an ordered ($v_a \neq 0$) phase, for all system sizes. Also clear is the fact that the transition point is less diffused for larger system sizes. Figure 3(b) shows the transition of v_a from zero to non-zero values as density is increased at fixed noise strength beyond a certain critical density ρ_c.

Vicsek et al. observed that the region over which the order parameter shows scaling increases with increasing system size (which can be seen from Figure 3(a)). They therefore concluded that barring some drastic *crossover effect*, in the thermodynamic limit $L \to \infty$, the transition to polar order must be continuous, with scaling of the order parameter with noise and density given by:

$$v_a \sim [\eta_c(\rho) - \eta]^\beta, v_a \sim [\rho - \rho_c(\eta)]^\delta, \tag{5}$$

where β and δ are critical exponents, and $\eta_c(\rho)$ and $\rho_c(\eta)$ are the critical noise and density respectively, in the thermodynamic limit. The authors estimated the critical exponent β from a plot of $\ln v_a$ vs $\ln[(\eta_c(L) - \eta)/\eta_c(L)]$ for different system sizes and fixed density $\rho = 0.4$. Note that for finite system sizes, the critical noise depends on L. β is simply the slope of the linear fit to this plot, with the scaling getting better and better for larger system sizes, as shown in Figure 4(a). Similarly the exponent δ is estimated from the slope of the linear fit to a plot of $\ln v_a$ vs $[(\rho - \rho_c(L))/\rho_c(L)]$ for $\eta = 2.0$ as shown in Figure 4(b). Vicsek et al. determined the critical exponents to be $\beta = 0.45 \pm 0.07$ and $\delta = 0.35 \pm 0.06$. They stated that the error estimates given are rather conservative, given the sensitivity of the critical exponents to the choice of critical noise and critical density for a system size L, which were estimated indirectly. The phase diagram of the flocking transition in the noise-density phase space would have a line of critical noise as a function of density. Consequently, you would expect β and δ to have the same value. The authors stipulated that even though their estimates of β and δ were not equal, this was an

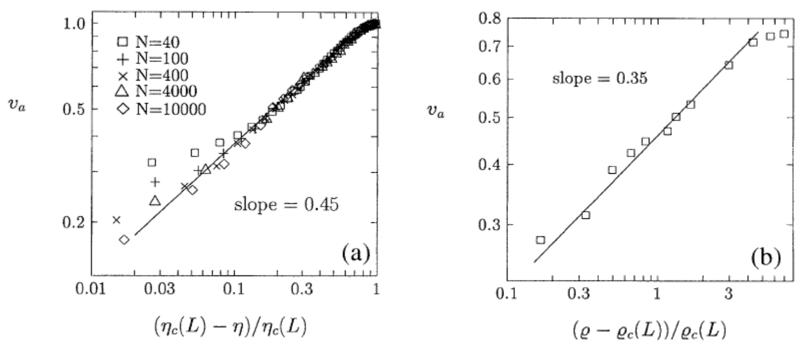

Figure 4: Determination if critical exponents (a) Plot of $\ln v_a$ vs $\ln[(\eta_c(L) - \eta)/\eta_c(L)]$ for different system sizes and fixed density $\rho = 0.4$ (b) Plot of $\ln v_a$ vs $[(\rho - \rho_c(L))/\rho_c(L)]$ for $\eta = 2.0$ and $L = 20$ [Picture taken from *Vicsek et al.* Phys. Rev. Lett. 75, 1226 (1995)]

artifact of strong finite size effects. As such there was no evidence as to why they should exclude the possibility of these exponents becoming equal in the thermodynamic limit, given their error bars.

2.3 Finite Size Scaling

If we accept that the flocking transition is indeed continuous, we can perform a finite size scaling estimation of the critical noise in the thermodynamic limit. We define a scaling function for the order parameter v_a for a finite system size L, which has the form:

$$v_a(t, L^{-1}) \sim t^{\beta} \Phi\Big(\xi_\infty L^{-1}\Big) = t^{\beta} \Phi\Big(t^{-\nu} L^{-1}\Big), \tag{6}$$

where ξ_∞ is the correlation length in the thermodynamic limit, and

$$t = \frac{\eta_c^{(\infty)}(\rho) - \eta(\rho)}{\eta_c^{(\infty)}(\rho)}, \tag{7}$$

where $\eta_c^{(\infty)}(\rho)$ is the critical noise strength for a given density in the thermodynamic limit. In the thermodynamic limit the correlation length is given by $\xi_\infty \sim t^{-\nu}$. As $L^{-1} \to 0$ for a fixed $t << 1$, we have the true critical behaviour and $v_a \sim t^{\beta}$. As $t \to 0$ for a finite size L, $\xi_\infty >> L$, and the transition appears rounded because the correlation length cannot exceed the system size. We can exploit this behaviour to figure out that the scaling function $\Phi(x)$ must be:

$$\begin{aligned} \Phi(x) &= t^{\beta} (t^{-\nu} L^{-1})^{\beta/\nu} \bar{\Phi}(t L^{1/\nu}), \\ &= L^{-\beta/\nu} \bar{\Phi}(t L^{1/\nu}), \end{aligned} \tag{8}$$

where $\bar{\Phi}$ is a new scaling function, which has the following asymptotic behaviour:

$$\bar{\Phi}(x) \underset{x\to\infty}{\sim} x^{\beta}, \tag{9}$$

$$\Phi(x) \underset{x\to 0}{\sim} A \ (const.). \tag{10}$$

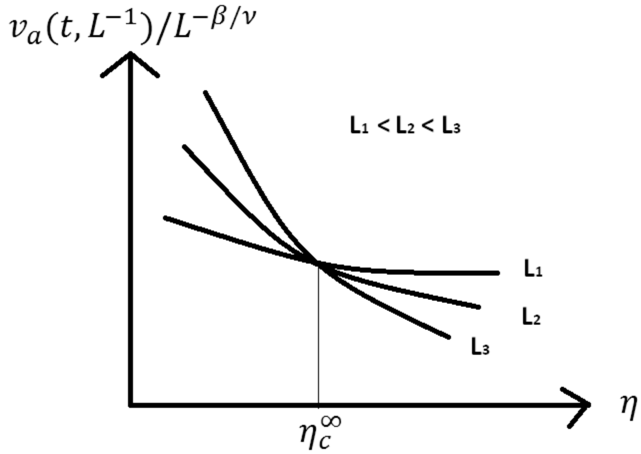

Figure 5: Cartoon picture of determination of critical noise in the thermodynamic limit via finite size scaling.

We can then expand around small t to write

$$\frac{v_a}{L^{-\beta/\nu}} = A + (tL^{1/\nu})^\beta + O(t^2). \tag{11}$$

At the critical noise strength in the thermodynamic limit, we have $t = 0$. A plot of $v_a/L^{-\beta/\nu}$ vs η must coincide at the same point $\eta = \eta_c^\infty(\rho)$ for all system sizes, and we can therefore calculate $\eta_c^\infty(\rho)$ from this plot (Figure 5).

The authors estimated the critical noise strength for $\rho = 0.4$ to be $\eta_c^\infty = 2.0 \pm 0.05$.

2.4 Model Summary

To summarize the findings of Vicsek et al., they observed that a collection of self-propelled particles, subject to a polar alignment interaction described by the simple rules of the Vicsek model, undergoes a "flocking transition" from an isotropic to a polar ordered phase, when density is increased above or noise decreased below a critical value. They concluded that the transition was continuous, and estimated the critical exponents for the order parameter (average velocity) scaling. They also estimated the critical noise strength in the thermodynamic limit for a fixed density, via finite size scaling.

2.5 Continuum Theory

A continuum dynamical model for the flocking transition was proposed by Toner and Tu [9] later in 1995, which encompasses a much larger universality class of polar alignment interactions, including that of the Vicsek model. They succeeded in calculating the scaling exponents for the order parameter exactly in two dimensions, and showed that the continuous theory predicts a broken continuous symmetry (i.e. a second order phase transition) in 2d. Here, let us draw the reader's attention to the fact that this is very different from equilibrium systems, which cannot exhibit a broken continuous symmetry at a finite temperature for $d \leq 2$, according to the *Merwin-Wagner theorem* [5]. A discussion on the continuum theory of flocking is however beyond the scope of this review, and the interested reader is directed to Toner and Tu's paper for a detailed analysis.

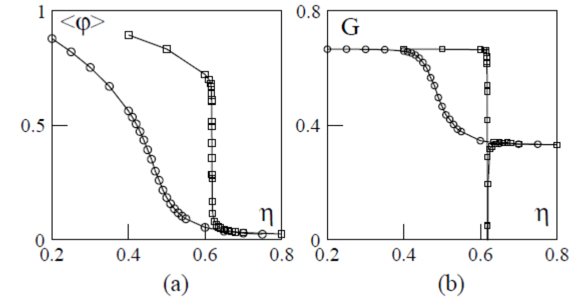

Figure 6: Comparison between the original VM (circles), and the model with vectorial noise (squares), $v_0 = 0.5, L = 32, \rho = 2$. (a) Order parameter vs noise and (b) Binder cumulant vs noise. [Picture taken from *Chaté et al.* Phys. Rev. Lett. 92, 025702, (2004)]

3 Discontinuous Transition To Polar Order

In 2004, about ten years after Vicsek et al. proposed their model, Gregoiré and Chaté showed that the transition to polar order in the VM in actually discontinuous [2], and that the apparent continuous nature of the transition was merely an artifact of strong finite size effects. Their analysis of the nature of the transition was based on computing the Binder cumulant:

$$G = 1 - \frac{\langle \phi^4 \rangle}{3\langle \phi^2 \rangle^2}, \tag{12}$$

where ϕ is the order parameter relevant to the transition, and $\langle ... \rangle$ denotes a spatial average. For a first order transition, the Binder cumulant dips to negative values at the critical value of the tuning parameter (density/noise in our case). For the flocking transition, the order parameter is the average velocity defined by equation 4.

The arguments made by Chaté and collaborators in favour of the 1st order nature of the flocking transition were twofold. Firstly, they modified the way noise was incorporated in the system by switching from a picture of "angular noise" (given in equation 2) to the following picture of a "vectorial noise":

$$\theta_j^{t+1} = arg\Big[\sum_{j \sim k} e^{\iota \theta_k^t} + \eta n_j^t e^{\iota \xi_j^t}\Big], \tag{13}$$

where the sum is over the neighbours of the j^{th} particle in a unit radius, η is the strength of noise, n_j^t is the number of neighbours of the j^{th} particle at time t, and ξ_j^t is a random number. The motivation behind vectorial noise is that an error could be made in aligning with the calculated average direction, as opposed to making an error in calculating the new direction itself. Figures 6(a) and (b) show plots of the order parameter $\phi = v_a$ and the Binder cumulant G respectively, as a function of noise strength, for the original Vicsek model (circles) and with the introduction of vectorial noise (squares). These plots have been taken from Chaté's paper. It is clear that the whereas the transition to polar order seems continuous in the original VM, for the same system size, the model with vectorial noise has a first order transition. This is indicated both by the sudden drop in the order parameter value above a certain noise, as well as the dropping of the Binder cumulant to negative values. Although introducing a vectorial noise changes the nature of the alignment interaction significantly, the authors argued that there was no convincing mechanism by which the order of the transition could be modified simply by

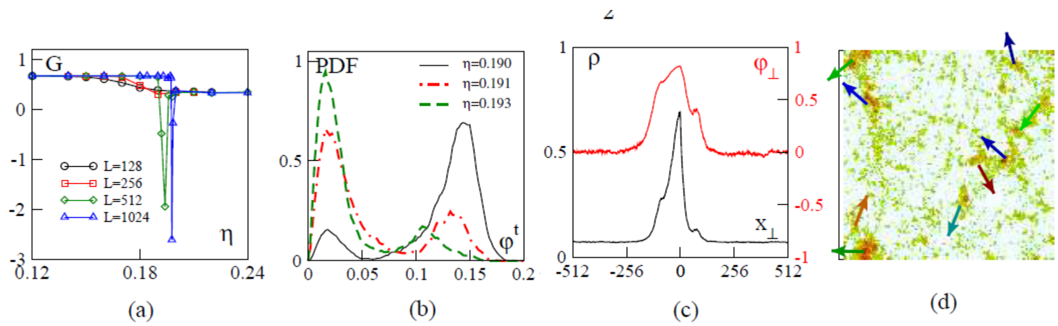

Figure 7: First order transition in the original VM at $\rho = 1/8$. (a): G vs η at various system sizes. (b): Probability distribution function of the order parameter ϕ near the transition point. (c): transverse density (bottom curve) and order parameter profile (top curve) in the ordered phase ($L = 1024, \rho = 0.18$). (d) Snapshot of coarse-grained density field in disordered phase at threshold, $\rho = 2, L = 256$. The arrows indicate the direction of motion of dense, ordered regions. [Picture taken from *Chaté et al.* Phys. Rev. Lett. 92, 025702, (2004)]

changing the way noise was added to the system.

This somewhat weak argument was augmented however by looking at larger system sizes for the original VM itself, which were not previously investigated. The authors were able to show that for large enough system sizes, even the original VM displays a discontinuous onset of order. Figure 7(a) shows the Binder cumulant vs noise in the original VM at a fixed density $\rho = 1/8$, and the first order nature of the transition at large system sizes is evident. Further, Figure 7(b) shows the probability distribution of the order parameter near the threshold noise, and one can see that the distribution is bimodal for all noise strengths near the transition, with no intermediate unimodal profile. This again indicates that the order parameter changes discontinuously from zero to non-zero values.

The authors further show that at the onset of order, the ordered phase consists of a high density ordered band moving in a disordered gas like background (Figure 7(c)). It's only at much higher densities (or low noise strengths) that the homogeneous ordered phase postulated by Toner and Tu [9] is encountered, something we will talk about in more details in the next section.

The first order nature of the flocking transition has been the subject of much controversy. It is now accepted that the the discontinuous nature of the onset of polar order is fluctuation driven [3][6]: at the threshold of the transition, the homogeneous ordered phase is unstable to a long wavelength instability with propagating unstable modes $k < k_0$. This results in the appearance of high density ordered bands traveling in a disordered background right at the transition boundary. The transition appears continuous for system sizes $L < L_0 \sim 1/k_0$, because such small systems are unable to sustain the unstable modes. These unstable modes however become manifest at larger system sizes, changing the order of the transition from continuous to discontinuous.

4 The Flocking Transition As A Liquid-Gas Transition

In 2015 Chaté et al. [7] showed that the flocking transition is better understood as a liquid-gas transition rather than an order-disorder transition, if we consider the dense ordered phase to be a "liquid", and the low density disordered phase to be a "gas". The reason for such a classification is that, on increasing the density beyond a critical value, high density ordered bands appear, which move in a low

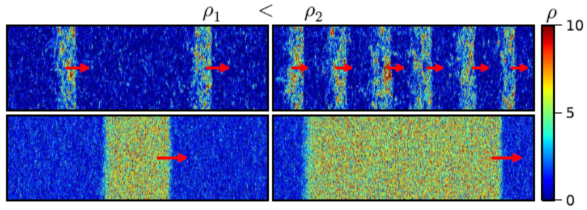

Figure 8: Top: Microphase separation in the Vicsek model. $\eta = 0.4$, $v_0 = 0.5$, $\rho_1 = 1.05$ (left), $\rho_2 = 1.93$ (right). Bottom: Phase separation in the active Ising model. $D = 1$, $\epsilon = 0.9$, $\beta = 1.9$, $\rho_1 = 2.35$ (left), $\rho_2 = 4.7$ (right). System sizes 800. Red arrows indicate the direction of motion.[Picture taken from *Chaté et al.* Phys. Rev. Lett. 114, 068101 , (2015)]

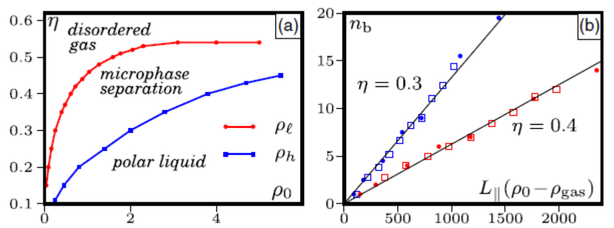

Figure 9: (a) Phase diagram of the Vicsek model. The binodals $\rho_l(\eta)$ and $\rho_h(\eta)$ mark the low density and high density limits of the coexistence region respectively. (b) Number of bands n_b vs $L_{||}(\rho_0 - \rho_{gas})$ varying either the excess density for system size 2000×100 (squares) or the system size along the direction of propagation (dots) for $\rho_0 = 0.6, (\eta = 0.3)$ or $\rho_0 = 1.2, (\eta = 0.4)$. The straight black lines are guides for the eyes. [Picture taken from *Chaté et al.* Phys. Rev. Lett. 114, 068101 , (2015)]

density disordered background. The homogeneous ordered phase postulated both by the VM and the continuum theory [9] is observed only after a second transition at much higher densities, or lower noise. The intermediate density/noise regime then consists of coexistence between the ordered and disordered phases, just like in a liquid-gas phase transition. However in the case of the flocking transition, instead of a single liquid domain coexisting with the gas phase, multiple ordered high-density (liquid) bands are observed to move in a "gaseous" background in the coexistence phase. Thus, it makes sense to think of the flocking transition as a liquid-gas phase transition with *micro-phase separation*, as opposed to the complete phase separation observed in real fluids in the coexistence regime.

The bands in the inhomogeneous phase of the flocking transition are oriented orthogonal to their direction of motion, with the particles aligned along the short axis (Figure 8(a)). The origin of these bands is ascribed to a long wavelength instability that was briefly discussed in the previous section, and will not be explored further in this paper. Instead we focus on the phase diagram of the model in the noise-density phase space. This phase diagram is given in Figure 9(a), which has been taken from [7]. We observe three distinct phases - a disordered "gas" at high noise strengths and low densities, an ordered "liquid" at high densities and low noise, and an intermediate inhomogeneous region of gas-liquid coexistence. In the thermodynamic limit, the two binodals $\rho_l(\eta)$ and $\rho_h(\eta)$ separate the inhomogeneous phase from the homogeneous phases. If the system is quenched from either homogeneous phase into the coexistence region, near the phase boundaries nucleation and annihilation of bands is observed, while deep in the inhomogeneous phase one observes spinodal decomposition. From Figure 9(a) we notice a second distinguishing feature of the flocking phase diagram from that of a real fluid - the critical point of

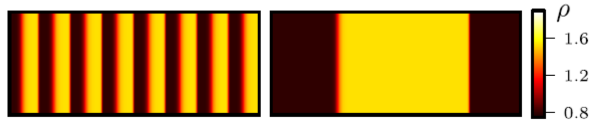

Figure 10: Density field in the PDEs after integration over $t = 10^5$. Left: scalar PDE (AIM), ordered initial condition with a periodic perturbation. Right: vectorial PDE (VM), disordered initial condition. Both profiles are for the same range of parameters. [Picture taken from *Chaté et al.* Phys. Rev. Lett. 114, 068101 , (2015)]

the phase transition is at infinity. This is because there is no continuous way to go from the disordered gas to the polar liquid at any noise strength or density.

4.1 Band Structure

Starting with random initial conditions, with average density ρ_0 and noise strength η corresponding to the coexistence regime, bands form rapidly. They are randomly spaced to begin with, but then relax to regular spacings after sufficiently long time. Increasing ρ_0 at constant noise in this regime does not affect the density of the gaseous background ρ_{gas}, nor the speed c of the bands, for both finite and infinite systems. Increasing ρ_0 simply increases the number of bands n_b proportional to $L_{||}(\rho_0 - \rho_{gas})$, as can be seen in Figure 9(b). In the thermodynamic limit, the bands have fixed profiles independent of the average density. However in finite systems, unless the excess density $S(\rho_0 - \rho_{gas})$, where S is the surface area, is a multiple of the excess density in each band, a new band cannot be nucleated. The excess density then causes the bands to deform slightly, keeping ρ_{gas} and c constant.

4.2 Comparison With An Active Ising Model (AIM)

At this point, let us discuss similarities of the VM with the recently proposed active Ising Model (AIM)[8] by Solon et al., which also exhibits a liquid-gas type phase transition. Very briefly, the AIM consists of N spins in a d dimensional lattice, which hop to their neighbouring sites with symmetric hopping rates in all directions but one. This asymmetry in hopping rates mimics self-propulsion. Also, the spins are not constrained by any exclusion principle, and more than one spin can occupy the same lattice point. Three different phases are again obtained for this model, a uniform disordered phase at high temperatures, a uniform ordered phase at low temperatures, and a phase separated profile at intermediate temperatures with a single high density ordered band traveling in a low density background.

When we compare the AIM to the VM, the first thing to note is that the flocking transition involves the spontaneous breaking of a *discrete* symmetry in the former, and a *continuous* symmetry in the latter. Also, although the onset of order in both models can be described as a liquid-gas phase transition, the AIM exhibits complete phase separation in the coexistence region, whereas the VM shows microphase separation (Figure 8). Increasing the average density in the coexistence region of the AIM simply widens the single liquid domain, whereas in the VM, new bands are nucleated.

Hydrodynamic continuum theories can be written down for both models, with one distinguishing feature: the order parameter for the VM is a vector (polarization), whereas the order parameter for the AIM is a scalar (magnetization). Thus, it might be the symmetry of the order parameter that causes such stark differences in the nature of the inhomogeneous phases of the two models. However, very interestingly, it was demonstrated [1] recently that such continuum theories admit both traveling bands as well as phase separated profiles as solutions, irrespective of the scalar or vectorial nature of the order parameter. This implies that both the VM and the AIM can potentially have both kinds of inhomogeneous phases, and yet, each model picks only one kind of solution every single time. In [7] Chaté et al. checked whether this difference in the inhomogeneous phases was due to the different *stability* of each type of solution in the two models. They looked at scalar and vectorial versions of the same minimal partial differential equation to study the dynamics of the order parameter in the AIM and the VM respectively. They demonstrated that both the scalar partial differential equation (sPDE) for the AIM and the vectorial partial differential equation (vPDE) for the VM admits both traveling bands as well as phase separated profiles in the inhomogeneous phase. Which kind of solution is actually selected depends solely on the initial conditions. Figures 10(a) and (b) show traveling bands in the AIM and phase separation in the VM respectively, which proves that the symmetry of the order parameter alone cannot explain the difference in the inhomogeneous profiles observed in the agent based models. The authors then investigated a stochastic version of this minimal continuum theory, by including a scalar and vectorial Gaussian white noise into the sPDE for the AIM and the vPDE for the VM respectively. The introduction of the stochasticity recovers the actual inhomogeneous profiles observed in the agent based models - traveling bands in the VM and phase separated profiles in the AIM. Starting from a large traveling liquid domain in a gaseous background as the initial condition, the AIM sustains this phase separated profiled, whereas in the VM, the liquid domain breaks up into an array of traveling bands. Similarly, starting with traveling bands, initially both models exhibit merging, but in the VM this process stops while in the AIM the merging continues till there is only one large liquid domain.

We then understand that the difference between the inhomogeneous phases of the VM and the AIM are a result of a stochasticity driven pattern selection mechanism. Giant density fluctuations are observed in the liquid phase of the VM, which breaks up large liquid domains, and prevents band coarsening. On the other hand in the AIM, density fluctuations are normal, and the inhomogeneous profile is phase separated.

5 Summary

In this essay we reviewed the Vicsek model, which was the first agent based model that described flocking in active matter, and which is attractive because of it's minimal nature and numerical tractability. We looked at Vicsek's original paper and discussed their main results, which was that a system of self-propelled particles subject to a polar alignment mechanism undergoes a continuous phase transition from an isotropic phase to a polar ordered phase. We then talked about the work of Chaté and collaborators which proved that the flocking transition was actually first order, because of a long wavelength instability that destabilizes the homogeneous polar ordered phase at the threshold of the transition. We also discussed how the flocking transition must be treated as a liquid-gas phase transition because it has an intermediate regime of phase coexistence. We compared the Vicsek Model to the recently proposed active Ising model, which also exhibits a liquid-gas type phase transition, but has a different phase coexistence profile. Whereas the Vicsek model shows microphase separation, the active Ising model exhibits completely phase separated profiles in this intermediate regime. We learnt that the difference in the coexistence phases of the two models is stochasticity driven. The Vicsek model, with a vector order parameter, has an ordered phase which is susceptible to giant number fluctuations, whereas the active Ising model, with a scalar order parameter, only has normal fluctuations. Thus even though both models can potentially have both complete or micro phase separation in the coexistence region, stochasticity serves as a pattern selection mechanism, which results in giving these models their characteristic inhomogeneous profiles.

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