# ASPECTS OF THE EXACT RENORMALIZATION GROUP

**Souvik Dutta**

## CONTENTS



## ABSTRACT

Various aspects of the Exact Renormalization Group (ERG) are explored. In § I, we start with a review of the concepts underlying the framework, paying special attention to developing an intuitive picture of rescalings in an ERG algorithm (§ II). We proceed to uncover Polchinski's ERG equation (and its related cousins) in § III and in the process, obtain an interpretation of a continuum blocking function. In § IV, we attempt at solving the flow equation using non-perturbative truncations and encounter the ever-so-illuminating hurdles of handling such calculations. We conclude with hunting for fixed points (some really cool techniques!) for ERGEs which don't have analytic solutions.

## I. INTRODUCTION

The physical intuition behind the Exact Renormalization Group (ERG) is a very simple observation : the physics of a system depends on the scale used to describe it. In momentum space, it involves iteratively *integrating out* (course-graining) high energy *modes* (degrees of freedom) of the system.

ERG has as its central ingredient, the Wilsonian effective action. The action $S_{\Lambda_0}(\{g_i^0\})$ for a system at a *bare scale* $\Lambda_0$ encodes the kinds of interactions and the strengths of the couplings $\{g_i^0\}$. Upon (at least formally) integrating out the degrees of freedom between $\Lambda_0$ and a lower *effective scale* $\Lambda < \Lambda_0$, we are left with the action $S_\Lambda(\{g_i\})$. This is the *Wilsonian effective action* at the scale $\Lambda$. The ERG (or flow) equation governs the behavior of $S_\Lambda$ under *infinitesimal* changes of scale $\Lambda \to \Lambda'$. As we shall see (e.g. Eq (5)), it has the basic form

$$-\Lambda\partial_\Lambda S_\Lambda[\varphi] = \ldots \tag{1}$$

where $\{\varphi\}$ is a set of fields in the action. That there exist methods to *exactly* solve the flow equation renders ERG indispensible. We will look at a class of solutions using *non-perturbative truncations*, pioneered by Hasenfratz. We shall go through many of our arguments for a lattice model, since it is more intuitive that its continuum counterpart, and since the lessons learnt carry over to the continuous ERG limit.

## II. QUALITATIVE ASPECTS

As it turns out, the correct thing to do to understand renormalization, is to add a second ingredient to the ERG transformation (on top of the coarse-graining): a *rescaling*. Let us consider a lattice, with a spin $\uparrow$ or $\downarrow$ at each site and spacing $a$. Let us suppose that we coarse-grain over $n \times n$ blocks, thus replacing them by a single *Kadanoff-blocked* spin. As a result, the distance between blocked spins is $na$. If we wish to compare the descriptions of the original and the coarse-grained system, we should rescale $na \to a$; this restores the cutoff $\Lambda \to \Lambda_0$. This is shown in Fig 1.

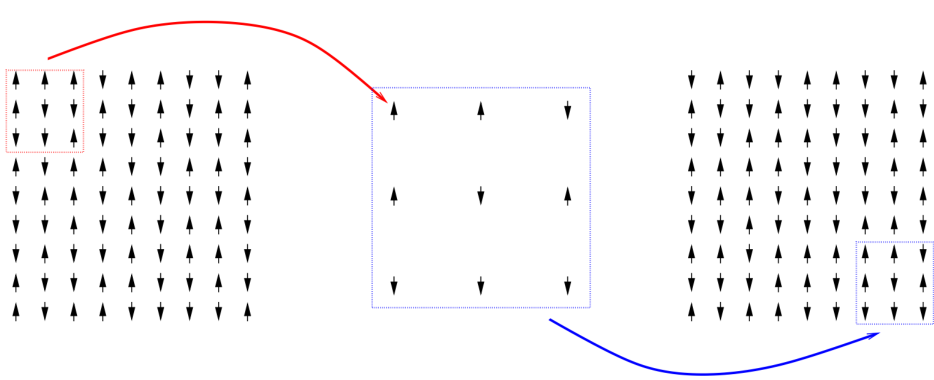

FIG. 1. The two steps of coarse-graining (majority rule) and rescaling are performed on the lattice. The right inset is to be understood as fitting the coarse-grained *block* by zooming out and bringing in spins that were outside the left inset.

To **note** is that we shall only require *locality* (averaging only over local patches), and the *invariance* of the partition function $\mathcal{Z}$. Also, nearest neighbor (NN) interactions get *spoiled* to NNN, NNNN, . . . . We shall discuss all this more in the following section(s).

The coupling (phase) space is a space labelled by $\{g_{\text{NN}}, g_{\text{NNN}}, \ldots\}$, and by doing the RG procedure, we hop around in this space. This space can have multiple fixed points (FP) - the (usually non-compact) manifold generated by flows *into* a particular FP is called the *critical manifold.* As expected, the critical manifold is spanned by irrelevant deformations around the FP (Fig 2).

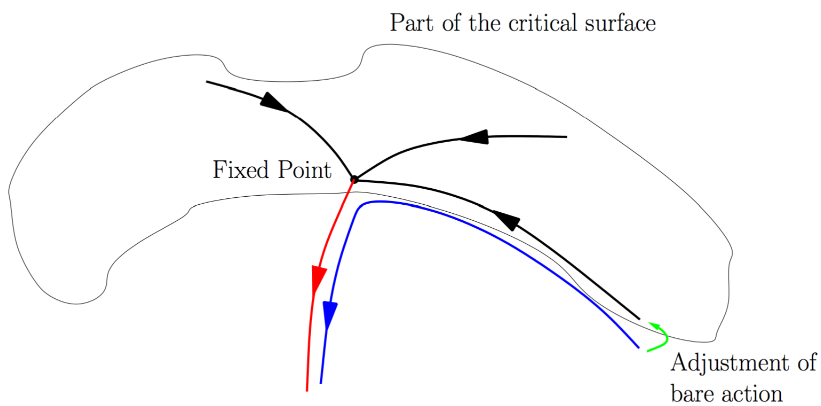


FIG. 2. (Part of) the critical manifold generated by irrelevant RG flows, wrt the shown FP. The red line emanating from the fixed-point is called a **renormalized trajectory**. The blue line shows a flow which starts just off the critical surface. By *adjusting* the bare action, this flow can be tuned towards the critical surface.

We shall start by solving the ERG equation to determine the spectrum of FPs, and the set of *renormalized trajectories* supported between them. If we find a FP, we linearize the ERG equation about the fixed-point to determine whether the various operators $\{\mathcal{O}_i\}$ are relevant, irrelevant or marginal. (To make contact with literature, we define the RG time $t = -\ln \Lambda/\mu$; in the continuum limit, as we flow from the UV to IR, the RG time goes from $t_{\text{UV}} = -\infty$ to $t_{\text{UV}} = +\infty$.) Given a bare action $S_\star[\varphi]$ that we have *solved for* near a FP, the linearized flow equation about that FP is of the form

$$S_t[\varphi] = S_\star[\varphi] + \sum_{\alpha_i} e^{\lambda_i t}\mathcal{O}_i[\varphi] \tag{2}$$

Of course, the physics is governed by operators $\{\mathcal{O}_i\}$ with $\lambda_i \geq 0$, i.e. if they are relevant or marginal. We shall now start looking at flow equations for the scalar field theory.

## III. FLOW EQUATIONS FOR SCALAR FIELD THEORY

### A. ERG equation : beauté dans simplicité

We work in $d$-dimensional *Euclidean* space, and shall closely follow the exposition in [1]. To begin with, we should clarify the requirement $\mathcal{Z} \xrightarrow{\text{ERG}} \mathcal{Z}$,

$$\mathcal{Z} = \int_{\Lambda_0} \mathcal{D}\Phi e^{-S[\Lambda_0]} = \int_{\Lambda} \mathcal{D}\Phi e^{-S[\Lambda]} \tag{3}$$

since this philosophy is central to our arguments. The partition function $\mathcal{Z}$ encodes the physics, but knows nothing about the choice of a scale. However, the converse is not true : the Wilsonian effection action *does* know about universal (scale/frame-independent) quantities; changing $\mathcal{Z}$ would change the physics itself, not just our *description* of it.

In a rather abstract way, (3) is strung into the ERG equation as (see [2])

$$-\Lambda\partial_\Lambda e^{-S_\Lambda[\varphi]} = \int \frac{\delta}{\delta\varphi(x)}\left(\Psi_\Lambda(x)e^{-S_\Lambda[\varphi]}\right) d^d x \tag{4}$$

for some choice of $\Psi_\Lambda(x)$. A few comments are as follows. The invariance of $\mathcal{Z}$ follows from a functional integration $\int(\ldots)\mathcal{D}\varphi$ of (4); the LHS is just the RG flow of $\mathcal{Z}$, the RHS vanishes from being a total derivative. The functional $\Psi[\varphi(x)]$ is the continuum analogue of Kadanoff blocking; to respect brevity, we shall justify this using an example. The **flow equation** can be read off from (4) as

$$-\Lambda\partial_\Lambda S_\Lambda[\varphi] = \int\left(\frac{\delta S_\Lambda}{\delta\varphi(x)}\Psi_\Lambda(x) - \frac{\delta\Psi_\Lambda(x)}{\delta\varphi(x)}\right) d^d x \tag{5}$$

and has the form promised in (1).

It may be obvious (we'll assume not, and spell it out) that (5) can be obtained from an infinitesimal *field* redefinition $\varphi'(x) = \varphi(x) - \delta t\Psi(x)$. Then,

$$\mathcal{Z} = \int \mathcal{D}\varphi' e^{-S_\Lambda[\varphi']} = \int \mathcal{D}\varphi\ e^{-S_\Lambda[\varphi]+\mathcal{G}[\Psi]S_\Lambda[\varphi]\delta t} + \mathcal{O}\left((\delta t)^2\right), \qquad \delta t = -\delta\Lambda/\Lambda$$

where the Wigner function $\mathcal{G}[\Psi]$ is the integrand in (5); the flow equation (5) follows in the limit $\delta\Lambda \to 0$. Warning : the measure Jacobian $\mathcal{J}(\varphi';\varphi)$ is non-trivial, more in [3].

### B. Flowing à la Polchinski

*This section is (unfortunately) plagued with new notation, so we refer the reader to the appendix.* Let us begin with a bare action

$$S_{\Lambda_0}[\varphi] = \frac{1}{2}\varphi\bullet\Delta^{-1}\bullet\varphi + S^{\text{int}}_{\Lambda_0}[\varphi] \tag{6}$$

where $\Delta$ in the kinetic term is the standard propagator $1/p^2$ and $S^{\text{int}}_{\Lambda_0}[\varphi]$ contains everything else, *including* the mass term. The obvious first step is to regularize the propagator :

$$\Delta_{\text{UV}} = \frac{C_{\text{UV}}(p,\Lambda_0)}{p^2}, \quad \widehat{\Delta}_{\text{UV}} \equiv -\Lambda\partial_\Lambda\Delta_{\text{UV}}. \tag{7}$$

Without further adieu, let us write down the Polchinski equation [4]:

$$-\Lambda\partial_\Lambda S^{\text{int}}_\Lambda = \frac{1}{2}\frac{\delta S^{\text{int}}_\Lambda}{\delta\varphi}\bullet\widehat{\Delta}_{\text{UV}}\bullet\frac{\delta S^{\text{int}}_\Lambda}{\delta\varphi} - \frac{1}{2}\frac{\delta}{\delta\varphi}\bullet\widehat{\Delta}_{\text{UV}}\bullet\frac{\delta S^{\text{int}}_\Lambda}{\delta\varphi}. \tag{8}$$

The RHS of (8) appears nasty, and without an understanding of its derivation, is opaque. For starters, we need to do two things : by comparing with Eq (4),

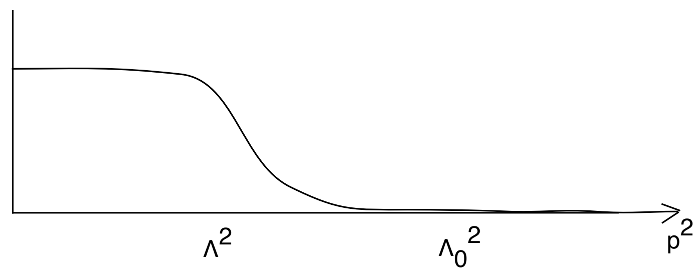


FIG. 3. Behavior of the UV cut-off function $C_{\rm UV}$ with momentum scale.

- determine the appropriate $\Psi_\Lambda(x)$ that gives rise to the Polchinski equation (8), and
- interpret $\Psi_\Lambda(x)$ as the "blocking" function.

With little justification, let us start with the ansatz

$$\Psi_\Lambda(x) = \frac{1}{2}\widehat{\Delta}(x,y)\frac{\delta\Sigma}{\delta\varphi(y)}, \qquad \Sigma = S - 2\widehat{S} \tag{9}$$

where $\hat{S}$ is a *non-universal* input called the **seed action**, and controls the precise trajectory of the flow. A technical requirement for $\widehat{S}$ is to *at least* share the symmetries of $S$; more comments later.

The flow equation (5) for the "blocking" function in (9) gives

$$-\Lambda\partial_\Lambda S_\Lambda = \frac{1}{2}\frac{\delta S_\Lambda}{\delta\varphi}\bullet\widehat{\Delta}\bullet\frac{\delta\Sigma}{\delta\varphi} - \frac{1}{2}\frac{\delta}{\delta\varphi}\bullet\widehat{\Delta}\bullet\frac{\delta\Sigma}{\delta\varphi} \tag{10}$$

which is starkly similar to (8). If we rewrite $S$ and $\widehat{S}$ as

$$S = \frac{1}{2}\varphi\bullet\Delta^{-1}\bullet\varphi + S^{\rm R}[\varphi], \quad \widehat{S} = \frac{1}{2}\varphi\bullet\Delta^{-1}\bullet\varphi + \widehat{S}^{\rm R}[\varphi],$$

some work shows that in fact

$$-\Lambda\partial_\Lambda S^{\rm R}_\Lambda = \frac{1}{2}\frac{\delta S^{\rm R}}{\delta\varphi}\bullet\widehat{\Delta}\bullet\frac{\delta\Sigma^{\rm R}}{\delta\varphi} - \varphi\bullet\Delta^{-1}\bullet\widehat{\Delta}\bullet\frac{\delta\widehat{S}^{\rm R}}{\delta\varphi} - \frac{1}{2}\frac{\delta}{\delta\varphi}\bullet\widehat{\Delta}\bullet\frac{\delta\Sigma^{\rm R}}{\delta\varphi} \tag{11}$$

where $\Sigma^{\rm R} = S^{\rm R} - 2\widehat{S}^R$. If we set the seed action $\widehat{S}^R \to 0$, we trivially recover the Polchinski action (8), with $S^{\rm R} \to S^{\rm int}$,

$$-\Lambda\partial_\Lambda S^{\rm int}_\Lambda = \frac{1}{2}\frac{\delta S^{\rm int}}{\delta\varphi}\bullet\widehat{\Delta}\bullet\frac{\delta\Sigma^{\rm int}}{\delta\varphi} - \frac{1}{2}\frac{\delta}{\delta\varphi}\bullet\widehat{\Delta}\bullet\frac{\delta\Sigma^{\rm int}}{\delta\varphi}. \tag{12}$$

Some comments follow. We did this song and dance about $\widehat{S}$, is to keep track of the freedom we have in choosing $\Psi_\Lambda(x)$; the ansatz in (9) completely gauge fixes this ambiguity. Only a particular choice of $\widehat{S}^R \to 0$ is the Polchinski equation, and so we have derived a *generalized Polchinski flow equation*. Henceforth, we shall drop the center term, but continue to write $\Sigma^{\rm int}$ instead of $S^{\rm int}$.

### C. Including the anomalous dimension

One of the things which makes QFT so rich is that quantum fields can acquire *anomalous dimensions*, which really means that the scaling dimension of the field is not equal to the canonical dimension. While a discussion can be found in standard QFT texts, we will be following the approach in [5].

As we have been advertising, the ERG procedure consists of two steps: a coarse-graining, followed by a rescaling. Traditionally, this latter operation is performed by considering an explicit dilatation (there are various equivalent ways of doing this) and computing its effect on the $S_\Lambda$. Equivalently, as in [5], we can instead rescale all quantities to dimensionless ones using the effective scale, $\Lambda$. That means,

$$\varphi(x) \to \Lambda^{(d-2)/2}\varphi, \qquad x \to \Lambda^{-1}x, \qquad \varphi(p) \to \Lambda^{-(d+2)/2}\varphi(p), \qquad p \to \Lambda p \tag{13}$$

where the last two rescalings follow from the first two by a Fourier transform. The exponents of $\Lambda$ follow from *dimensional analysis*. This section will contain partial treatment of this rescaling; the full-blown effect will be considered in § IV.

However, under an RG flow, we will have (e.g. for the kinetic term in $S$), in momentum space

$$\int \frac{d^dp}{(2\pi)^d}\varphi(-p,\Lambda_0)\Delta^{-1}\varphi(p,\Lambda_0) \longrightarrow \frac{1}{2!}\frac{1}{Z_\Lambda}\int \frac{d^dp}{(2\pi)^d}\varphi(-p,\Lambda)\Delta^{-1}\varphi(p,\Lambda) \tag{14}$$

where the extra factor of $Z_\Lambda^{-1}$ can be interpreted as a *field-strength renormalization* (FSR) $\varphi \to \sqrt{Z_\Lambda}\varphi$. When we crank the machinery further, we have extra terms cropping up as

$$\begin{aligned} -\Lambda\partial_\Lambda\big|_\varphi S_\Lambda &\longrightarrow -\Lambda\partial_\Lambda\big|_\varphi S_\Lambda + \frac{\gamma}{2}\varphi\cdot\frac{\delta}{\delta\varphi}S_\Lambda, \\ \frac{\delta}{\delta\varphi}\cdot\widehat{\Delta}\cdot\frac{\delta}{\delta\varphi} &\longrightarrow \frac{1}{Z_\Lambda}\frac{\delta}{\delta\varphi}\cdot\widehat{\Delta}\cdot\frac{\delta}{\delta\varphi} \end{aligned} \tag{15}$$

The anomalous dimension $\gamma \equiv \Lambda\partial_\Lambda \ln Z$ is the flow of the field strength renormalization $Z$, as expected. And it is here to stay, as a *necessary evil* of doing $\Lambda$-dependent rescalings of $x, \varphi(x)$.

We will now do something pretty crazy : to maintain sanity in our equations, we will set $Z_\Lambda \to 1$. We shall not provide a proof (for one, refer to [6]), but we have enough freedom stemming from our good ol' $\Psi_\Lambda(x)$ to do this; $\Phi_\Lambda(x)$ in fact has changed (see Eq (9)); from the flow equation (5), we have

$$\left(-\Lambda\partial_\Lambda + \frac{\gamma}{2}\varphi\cdot\frac{\delta}{\delta\varphi}\right)S_\Lambda^{\rm int} = \frac{1}{2}\frac{\delta S^{\rm int}}{\delta\varphi}\cdot\widehat{\Delta}\cdot\frac{\delta\Sigma^{\rm int}}{\delta\varphi} - \frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\widehat{\Delta}\cdot\frac{\delta\Sigma^{\rm int}}{\delta\varphi}. \tag{16}$$

which we should compare with Eq (12) : only the LHS has picked up an anomalous $\gamma$.

### D. Diagrammatics for the Action

It is often useful, both from the point of view of doing certain calculations and for getting a better feeling for the flow equation, to introduce a diagrammatic representation. We will not use it for calculations (such as for the $\beta$ function), and only provide a fleeting treatment. As is usual in field theory, $S$ has an expansion in the fields $\varphi$, the coefficients represent the coupling constants

$$S = \frac{1}{2!} S^{(2)} \varphi\varphi + \frac{1}{4!} S^{(4)} \varphi\varphi\varphi\varphi + \ldots \tag{17}$$

and represent the terms as shown below.

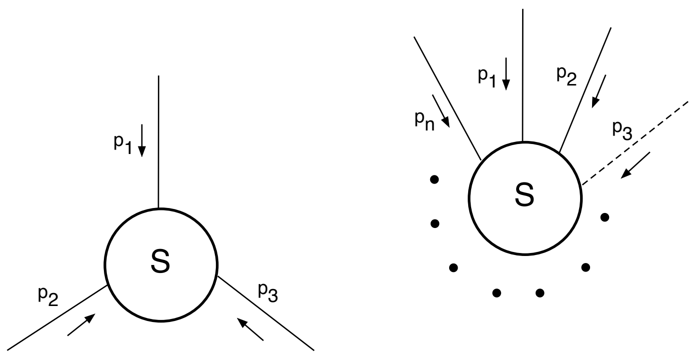


FIG. 4. The three-point vertex; Right : The generic $n$-point vertex

The vertices are denoted with fat circles instead of points to remind us that the diagrams contain *non-perturbative* information, in contrast with *Feynman diagrams*. Using the above examples, it is not too hard to write down the full flow equation (16), as in Fig (5).

Some comments follow : We have used the notation $[\ldots]^{(n)}$ for the full expansion in the fields. The operator $\varphi \cdot \delta_\varphi$ pulls down the factor of $n$ from $\varphi^n$ (in the parenthesis). The expression then is to be read as order-by-order in $n$. The $\delta_\varphi \cdot \widehat{\Delta} \cdot \delta_\varphi$ acting on $\Sigma^{\text{int}}$ gives rise to the loop diagram. Hence it is natural to associate the first term (tree diagram) on the RHS as a *classical* term and the second term (loop diagram) as a *quantum* term. The $\cdot$ in the propagator implies that internal momenta has been integrated over.

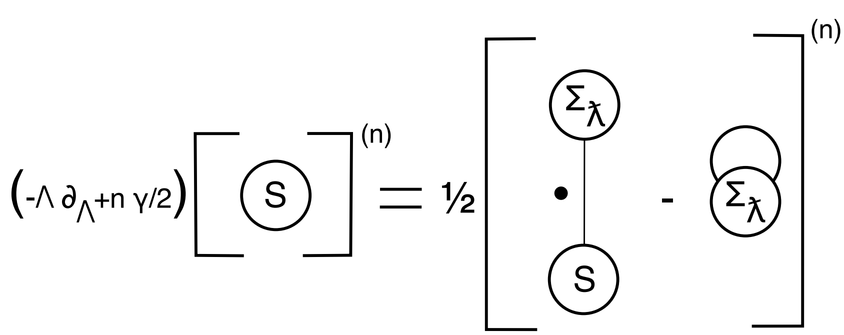


FIG. 5. The full flow equation written diagrammatically.

### E. Polchinski ERG as a Heat Equation

We will restrict ourselves to the Polchinski equation (8), but from the discussion it should be clear that the arguments hold true for its cousins (11) as well. To cast the Polchinski equation as a heat equation, we begin by defining the operator

$$\mathcal{Y} \equiv \frac{1}{2}\frac{\delta}{\delta\varphi} \cdot \widehat{\Delta} \cdot \frac{\delta}{\delta\varphi} \cdot \tag{18}$$

and make the $\Lambda$ dependence explicit

$$\mathcal{Y}_\Lambda = \frac{1}{2}\int_p \frac{\delta}{\delta\varphi(-p)}\frac{\Delta(p^2/\Lambda^2)}{p^2}\frac{\delta}{\delta\varphi(p)} \tag{19}$$

Then in terms of the RG-time derivative $\partial_t = \Lambda\partial_\Lambda$, the Polchinski equation (8) can be written as

$$-\partial_t e^{-S^{\rm int}[\varphi]} = -\dot{\mathcal{Y}} e^{-S^{\rm int}[\varphi]} \tag{20}$$

This has a structure of a heat equation, and is *not* just a random observation! This structure implies that in order for evolution with decreasing $\Lambda$ to correspond, in general, to a well-posed problem, we must take $\Delta'(p^2/\Lambda^2) < 0$ for $p^2/\Lambda^2 < \infty$. In particular, we must take $\Delta'(0) < 0$.

**Further comments :** The Wilsonian effective action flows under RG transformations through a Polchinski-like (11), (12) equation. However, there are other formalisms of writing down the effective action : a well-known example is the *Effective Average Action* $\Gamma^{\rm int}_\Lambda$, which flows as

$$\frac{\partial\Gamma^{\rm int}_\Lambda[\varphi_c]}{\partial\Lambda} = \frac{1}{2}{\rm Tr}\left[\frac{\partial\Delta^{-1}_{\rm IR}}{\partial\Lambda}\cdot\left(\Delta^{-1}_{\rm IR} + \frac{\delta^2\Gamma^{\rm int}_\Lambda}{\delta\varphi_c\delta\varphi_c}\right)^{-1}\right]. \tag{21}$$

## IV. SOLVING ERGE USING NON-PERTURBATIVE TRUNCATIONS

### A. Setting up the Problem

Let us revisit our expression of the Polchinksi-like cousin in (16)

$$\left(-\Lambda\partial_\Lambda + \frac{\gamma}{2}\varphi\cdot\frac{\delta}{\delta\varphi}\right)S^{\rm int}_\Lambda = \frac{1}{2}\frac{\delta S^{\rm int}}{\delta\varphi}\cdot\widehat{\Delta}\cdot\frac{\delta\Sigma^{\rm int}}{\delta\varphi} - \frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\widehat{\Delta}\cdot\frac{\delta\Sigma^{\rm int}}{\delta\varphi}$$

where we will choose the seed action $\widehat{S}$ to be simple

$$\widehat{S} = \frac{1}{2}\varphi\cdot\Delta^{-1}(p,\Lambda)\cdot\varphi, \quad \text{where } \Delta(p,\Lambda) = \frac{C_{\rm UV}}{p^2}$$

We also recall how to write down the terms $[S]^{(n)}$ as in Fig (5) :

$$S = \sum_{n=2}^{\infty}\left(\prod_{i=1}^{n}\int\frac{d^dp_i}{(2\pi)^d}\right)S^{(n)}(p_1,p_2\ldots,p_n)\varphi(p_1)\ldots\varphi(p_n)(2\pi)^d\delta^{(d)}(p_1+p_2+\ldots+p_n). \tag{22}$$

For the uninitiated reader, this is the sum over $n$-point vertices, with $n$ undetermined (integrated) momenta. The delta function ensures that momentum conservation. The action is non-perturbative, hence has a sum over all $n \geq 2$.

Now let us scale out the engineering dimensions. Recall that the anomalous dimension $\gamma$ was an artifact of field strength renormalization, as discussed in § III C. Under the $x$ and $\varphi(x)$ scalings (13), we have

$$\left(-\Lambda\partial_\Lambda + \frac{\gamma}{2}\varphi \boldsymbol{\cdot} \frac{\delta}{\delta\varphi}\right) S \longrightarrow \left(-\Lambda\partial_\Lambda + \frac{\gamma - (d+2)}{2}\varphi \boldsymbol{\cdot} \frac{\delta}{\delta\varphi}\right) S$$

and that for $p$ and $\varphi(p)$,

$$\int \frac{d^d p_i}{(2\pi)^d} \to \Lambda^d \int \frac{d^d p_i}{(2\pi)^d}, \quad \delta^{(d)}(p_1 + \ldots + p_n) \to \Lambda^{-d}\delta^{(d)}(p_1 + \ldots + p_n).$$

Note that the operator $\varphi \boldsymbol{\cdot} \partial_\varphi$ counts the *number* of fields. We are sweeping some details under the rug; adding everything gives us (we have skipped writing "int" in view of clarity)

$$(\partial_t + [\varphi]\Delta_\varphi + \Delta_\partial - d)\, S = \frac{\delta S}{\delta\varphi} \boldsymbol{\cdot} C'_{\mathrm{UV}} \boldsymbol{\cdot} \frac{\delta\Sigma}{\delta\varphi} - \frac{\delta}{\delta\varphi} \boldsymbol{\cdot} C'_{\mathrm{UV}} \boldsymbol{\cdot} \frac{\delta\Sigma}{\delta\varphi}. \tag{23}$$

We are warranted to make several comments. $[\varphi] = \gamma + (d-2)$ is the *complete* scaling dimension of $\varphi$, from FSR and from (13). $\Delta_\varphi$ counts the number of fields $\{\varphi\}$ as we just remarked above. This is the full flow equation for the Wilsonian effective action after rescalings have been taken into account.

### B. The Derivative Expansion

The ERGE in (23) is an integro-differential equation, contains *variational* derivatives and is non-linear. From our experience in PDEs, we know that we have a difficult problem at hand - more so since we also have to tackle problems with non small parameter (i.e. perturbation theory isn't applicable); this has been explored in [7]. As is expected, the exact solutions are known for very special cases.

An alternative is to re-write the action in terms of derivatives of the field

$$S_\Lambda[\varphi] \sim \int d^d x \left(V_\Lambda(\varphi) + \frac{1}{2}(\partial_\mu\varphi)^2 K_\Lambda(\varphi) + \mathcal{O}(\partial^4)\right) \tag{24}$$

where $V_\Lambda(\varphi)$ is a *local potential* not carrying any derivatives, and $\mathcal{O}(\partial^4)$ are clubbed as *non-local* terms. **Warning :** That this works *in practice*, is surprising! At any rate, we shall begin with a simple example : set $K_\Lambda(\varphi) \to 1$ and throw away the non-local terms, as in

$$S_\Lambda[\varphi] \sim \int d^d x \left(V_\Lambda(\varphi) + \frac{1}{2}(\partial_\mu\varphi)^2\right) \tag{25}$$

This is called the *local potential approximation*, see [8]. In contrast to (14), the potential term does not renormalize : the $\Lambda$ dependence of $\varphi$ is exactly canceled by the measure and the derivative.

### C. The Hasenfratz Projection Method

One way to *re-write* (25) is to take the action in (22)

$$S = \sum_{n=2}^{\infty} \left( \prod_{i=1}^{n} \int \frac{d^d p_i}{(2\pi)^d} \right) S^{(n)}(p_1, p_2 \ldots, p_n) \varphi(p_1) \ldots \varphi(p_n) (2\pi)^d \delta^{(d)}(p_1 + p_2 + \ldots + p_n)$$

and impose the conditions

$$\begin{aligned} S^{(2)}(p, -p) &= \frac{1}{2} p^2 + S^{(2)}(0, 0), \\ S^{(n>2)}(p_1, \ldots p_n) &= S^{(n>2)}(0, 0 \ldots, 0). \end{aligned} \tag{26}$$

Seeing this is crucial to our calculations : let us first define the action of a *projector* $P(x)$ on some *test* functional $G[\varphi]$ (for details see [9])

$$P(x) G[\varphi] = e^{x \partial / \partial \varphi(0)} \Big|_{\varphi = 0} \tag{27}$$

which has a nice *factorization* property

$$P(x) \Big( G_1[\varphi] G_2[\varphi] \ldots G_m[\varphi] \Big) = \Big( P(x) G_1[\varphi] \Big) \Big( P(x) G_2[\varphi] \Big) \ldots \Big( P(x) G_m[\varphi] \Big)$$

leading to the simple result

$$P(x) S[\varphi] = \sum_{n=2}^{\infty} S^{(n)}(0, 0, \ldots, 0) x^n \delta(0). \tag{28}$$

The delta function can be swept away by putting the system in a finite box.

### D. Projection of the flow equation

The remarkable feature of Eq (28) is that projecting the flow equation leads to a simple mixed PDE in the *local potential*. Let us re-produce the (re-scaled) ERG equation (23) for convenience :

$$\left( \partial_t + [\varphi] \Delta_\varphi + \Delta_\partial - d \right) S = \frac{\delta S}{\delta \varphi} \cdot C'_{\mathrm{UV}} \cdot \frac{\delta S}{\delta \varphi} - \frac{\delta}{\delta \varphi} \cdot C'_{\mathrm{UV}} \cdot \frac{\delta S}{\delta \varphi}.$$

where we have sent $\Sigma^{\mathrm{int}} \to S^{\mathrm{int}}$ by requiring $\widehat{S}^{\mathrm{int}} \to 0$. This is the Polchinski regime. For the *classical* term,

$$\begin{aligned} P(x) \frac{\delta S}{\delta \varphi} &= \sum_{n=2}^{\infty} S^{(n)}(p, 0, \ldots, 0) n x^{n-1} \delta(p) = V'(x, t) \delta(p) \\ \Longrightarrow \int_p \left( P(x) \frac{\delta S}{\delta \varphi} \right) C'_{\mathrm{UV}} \left( P(x) \frac{\delta S}{\delta \varphi} \right) &= C'_{\mathrm{UV}}(0) V'^2(x, t) \equiv -K_0 V'^2(x, t), \quad K_0 > 0 \end{aligned} \tag{29}$$

One can only admire the cleanliness. The *quantum* term is

$$\int_p C'_{\mathrm{UV}} \frac{\delta^2 S}{\delta \varphi(p) \delta \varphi(-p)} = \sum_{n=2}^{\infty} S^{(n)}(p, -p, 0, \ldots, 0) n(n-1) x^{n-2} = V''(x, t) \int_p C'_{\mathrm{UV}}(p) \equiv -I_0 V'' \tag{30}$$

Both $K_0 > 0$ and $I_0 > 0$ follow from §III E. Putting two-and-two together and recalling that LPA (25) corresponds to the anomalous dimension $\gamma = 0$, we have

$$\partial_t V(x,t) = I_0 V'' - K_0 V'^2 - \frac{d-2}{2} x V' + d\, V \tag{31}$$

In fact, we can try to make it simpler by a rescaling :

$$V \to \frac{I_0}{K_0} V, \qquad x \to \sqrt{I_0} x,$$

such that (31) becomes

$$\partial_t V(x,t) = V'' - V'^2 - \frac{d-2}{2} x V' + d\, V \tag{32}$$

This is nice because it is just a non-linear PDE, as promised; also it is manifestly independent of any cut-off function.

### E. Finding Fixed Points

The requirement of a fixed point $V_*$ is $\partial_t V = 0$. Then from (32),

$$V_*'' - V_*'^2 - \frac{d-2}{2} x V_*' + d V_* = 0 \tag{33}$$

To solve this, we need two boundary conditions. Let us make a simple choice

$$V_*'(0) = 0, \qquad V_*(0) = \lambda \in \mathbb{R} \tag{34}$$

where by $V_*'(0) = 0$, we demand an *even potential.* But the solution for $V_*$ is a $\lambda$-parametrized set of fixed points. That ain't correct : we know that in $d = 3$ scalar QFT, we have a Gaussian FP and a Wilson-Fisher FP as critical FPs. Something must go wrong to signal a breakdown for most solutions.

To see this, first we note that Eq (33) doesn't have an (at least easy!) analytic solution. We attempt to solve it numerically using our favorite method, using a seed value of $\lambda = V_*(0)$. The solution $V(x,t)$ must be analytic for all $x$. Numerically, we compute the value $x = x_c$ for which $V(x,t)$ diverges; only the solutions $V(x,t)$ for which $x_c \to \infty$ are acceptable.

It turns out that only for $V_*(0) = 0$ does $x_c \to \infty$ for $d = 4$. Another method is to hunt for $V(x)$ that directly gives us globally non-singular solutions. From that we can back-track the requisite boundary conditions. For e.g., $V(x) \sim x^2$ at large $x$ gives us a discrete set of fixed points - we will not provide the proof, and is left as an exercise for the reader.

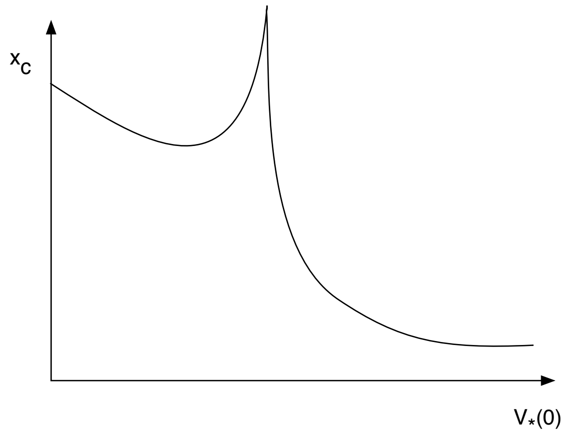


FIG. 6. Cartoon of $x_c$ at which $V(x)$ diverges plotted against the seed value $\lambda = V_*(0)$.

## V. APPENDIX

The ERG literature is usually plagued with notation that is remarkably compact. We provide a non-exhaustive list of notations.

- Integrals are written as

$$\int_x \equiv \int d^d x, \qquad \int_p \equiv \int \frac{d^d p}{(2\pi)^d}$$

The dimensions we are working in should be clear from context.

- Functional derivatives with respect to $\varphi(\bullet)$ are denoted by $\delta/\delta\varphi(\bullet)$ and satisfies

$$\frac{\delta\varphi(y)}{\delta\varphi(x)} = \delta^d(y-x), \qquad \frac{\delta\varphi(q)}{\delta\varphi(p)} = (2\pi)^d \delta^d(q-p)$$

- Both inner products and integrals over unfixed momenta are (unfortunately) written in an identical way - the meaning should be understood from context

$$A \bullet B \equiv \int_p A(p)B(-p) = \int_x A(x)B(x)$$
$$A \bullet K \bullet B \equiv \int_x \int_y A(x)K(x-y)B(y)$$

- Engineering dimensions are written is $[\ \bullet\ ]_c$, with the understanding

$$[L]_c = -1, \quad [\Lambda]_c = 1.$$

From this, the other dimensions are clear

$$[\varphi(x)]_c = \frac{d-2}{2}, \qquad [\varphi(p)]_c = -\frac{d+2}{2}.$$

[1] C. Bagnuls and C. Bervillier, "Exact renormalization group equations: An introductory review," Phys. Rept. **348** (2001) 91, hep-th/0002034.

[2] T. R. Morris, "A gauge invariant exact renormalization group. I," Nucl. Phys. **B 573** (2000) 97, hep-th/9910058.

[3] J. L. Latorre and T. R. Morris, "Exact scheme independence," *JHEP* **0011** (2000) 004, hep-th/0008123.

[4] J. Polchinski, "Renormalization And Effective Lagrangians," *Nucl. Phys.* **B 231** (1984) 269.

[5] T. R. Morris, "Derivative expansion of the exact renormalization group," *Phys. Lett.* **B 329** (1994) 241, hep-ph/9403340.

[6] R. D. Ball, P. E. Haagensen, J. I. Latorre, and E. Moreno, "Scheme Independence And The Exact Renormalization Group," Phys. Lett. **B 347** (1995) 80, hep-th/9411122

[7] T. R. Morris, "On truncations of the exact renormalization group," Phys. Lett. **B 334** (1994) 355, hep-th/9405190.

[8] J. F. Nicoll, T. S. Chang, and H. E. Stanley, "Approximate Renormalization Group Based on the Wegner-Houghton Differential Generator," *Phys. Rev. Lett.* **33** (1974) 540.

[9] A. Hasenfratz and P. Hasenfratz, "Renormalization Group Study Of Scalar Field Theories," Nucl. Phys. **B 270** (1986) 687 [*Helv. Phys. Acta*] **59** (1986) 833.