Renormalization group and sine-Gordon model

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Abstract

This essay describes the renormalization group approach to a 1D sine-Gordon model. This model is shown to be dual on the RG level to 2D X-Y spin model and exhibits the same Kosterlitz-Thouless transition near the critical point. We also discuss what role sine-Gordon model plays in studying boundaries of 2D topological materials, discuss its RG properties, and show how KT phase diagram analysis could possibly allow us to detect topological phases of matter experimentally.

1 Introduction

This term paper is mainly going to deal with $1+1D$ scalar field theory with the following action:

$$S = \int_{\mathcal{A}} d\tau dx \left[\frac{1}{v}(\partial_\tau \phi)^2 - v(\partial_x \phi)^2 \right] \tag{1}$$

where $\mathcal{A}$ is some area in $\tau - x$ space. Usually $\mathcal{A}$ is picked to be: $\mathcal{A} = \{0 < \tau < \beta, 0 < x < L\}$ with predefined β and L. We will mostly focus on studying perturbations to this theory of the following form:

$$\delta S = g \int_{\mathcal{A}} d\tau dx \cos(\beta\phi) \tag{2}$$

where parameter g is usually referred to as *the amplitude* and parameter β is the coupling constant. Turns out that this simple perturbation can yield a phase transition with highly non-trivial behavior near the critical point.

This simple scalar field action appeared in many contexts throughout the past fifty years and to this day it is one of the most thoroughly studied models. One might reasonably ask, what could be the reason to pick such a simplistic model as a topic for 2017 phase transitions' term paper. To answer this question, we need to turn our attention to the rapid theoretical and experimental progress in the study of 2-dimensional materials with nontrivial topological phases of matter. Turns out that the edge of such crystals support effective $1+1D$ bosonic theory. Coupling these materials to superconductors and/or turning on magnetic field allow us to introduce interaction terms to the underlying fermionic theory [6] which under bosonisation enter scalar field Lagrangian in the form of $\cos(\phi)$ terms. This provides us enough incentive to study sine-Gordon model from the RG point of view. More than that, this understanding of $1+1D$ scalar field theory as a bosonisation of a more fundamental fermionic theory living on the edge of a crystal means that we might also be interested in how do these perturbations affect spectrum of a fermionic theory both in the bulk and on the edge of the material and how does relevant and irrelevant perturbations differ from the point of view of electronic exications.

The structure of this term paper is as follows: we will begin by reviewing some preliminary information about correlation functions in scalar field theories. Then we'll discuss the general behavior of sine-Gordon model in infrared regime. We will see that there exist two different phases and that critical behavior is governed by renormalization of two parameters, g and β. Then we will explore the role such terms can play in the theory on the edge of FQH state and how these perturbations can be realized in practice.

2 Preliminaries

In this section we will quickly derive one of the most important formulas in scalar field theory closely following the logic presented in [1]. First, let us start with the partition function for the scalar field ϕ with the source η:

$$Z[\eta] = \int D\phi(\mathbf{x}) e^{-S - \int d\tau dx \eta(\mathbf{x})\phi(\mathbf{x})} \tag{3}$$

rewriting field and source in terms of their Fourier components we get:

$$Z[\eta] = \int \Pi_{\mathbf{p}} \phi(\mathbf{p}) \exp\left(-\frac{1}{2} \sum_{\mathbf{p}} \phi(-\mathbf{p})(vq^2 + v^{-1}\omega^2)\phi(\mathbf{p}) + \sum_{\mathbf{p}} \eta(-\mathbf{p})\phi(\mathbf{p}) \right) \tag{4}$$

where the variables are given

$$\mathbf{p} = (\omega, q), \quad \omega = \frac{2\pi n}{\beta}, \quad q = \frac{2\pi m}{L} \tag{5}$$

now, as we did numerous times in class, performing the shift of $\phi(\mathbf{p})$, we arrive at Gaussian integral in ϕ with additional term quadratic in source η:

$$Z = Z[\eta = 0] \times \exp\left(\frac{1}{2\mathcal{A}} \sum_{\mathbf{p}, \mathbf{p} \neq 0} \eta(-\mathbf{p}) G(\mathbf{p}) \eta(\mathbf{p}) \right) \tag{6}$$

where $G(\mathbf{p})$ is the usual Green's function:

$$G(\mathbf{p}) = \frac{1}{vq^2 + v^{-1}\omega^2} \tag{7}$$

this can be translated to position space as:

$$\frac{Z[\eta]}{Z[0]} = \exp\left(\frac{1}{2} \eta(\xi) G(\xi, \xi') \eta(\xi') \right) \tag{8}$$

where the Green's function in position space is given by [3]:

$$G(z, \bar{z}) = \frac{1}{4\pi} \ln\left(\frac{1}{z\bar{z} + a^2}\right), \quad \text{where} \quad z = \tau + i\frac{x}{v} \tag{9}$$

and a is the position space cutoff of the theory. So far we haven't done anything what wasn't discussed in class. However, right now we are going to make a particular choice of source function η that will allow us to compute very useful correlators:

$$\eta_0(\xi) = i\sum_{n=1}^{N} \beta_n \delta(\xi - \xi_n) \tag{10}$$

substituting this into (3) we get N-point correlation function of bosonic exponents:

$$\frac{Z[\eta_0]}{Z[0]} = \langle e^{i\beta_1\phi(\xi_1)} ... e^{i\beta_N\phi(\xi_N)} \rangle = \exp\left(-\frac{1}{2}\sum_{i,j} \beta_i G(\xi_i, \xi_j)\beta_j\right) \tag{11}$$

Substituting here the expression for Green's function (9), we can explicitly compute and analyze correlators of cosine operators (2) which basically have the form we are most interested in:

$$\cos(\beta\phi) = \frac{1}{2}e^{i\beta\phi} + \text{h.c.} \tag{12}$$

3 sine-Gordon model

Now let us study $1 + 1D$ scalar field theory with the action 1 and perturbations of the form 2. Let us state the following theorem: *A perturbation with scaling dimension d is relevant if $d > D$ and irrelevant if $d < D$*. Where D is the space-time dimension $D = 2$. Relevant perturbations are those whose influence grows on larger scales, while irrelevant perturbations fade out as we move away from critical point. Conformal field theory result [3] states that the operator

$$g\int_{\mathcal{A}} d\tau dx \cos(\beta\phi) \tag{13}$$

has the scaling dimension

$$d = \frac{\beta^2}{4\pi} \tag{14}$$

It means that for $d \equiv \frac{\beta^2}{4\pi} < 2$ this perturbation is relevant. Relevant perturbations can break conformal symmetry leading to the final correlation length. Let us observe how exactly this happens in sine-Gordon model.

3.1 Bosonic mass gap

One of the many ways to demonstrate that is to show that this theory describes a massive field. For simplicity, let's assume that $\beta \ll 1$ then we have, expanding cosine around the point $\phi = \pi/\beta$:

$$\cos(\beta\phi) \approx -1 + \frac{1}{2}\beta^2(\phi - \pi/\beta)^2 + ... \tag{15}$$

this turns our initial action into theory of massive scalar field:

$$S = \int_{\mathcal{A}} d\tau dx \left[\frac{1}{v}(\partial_\tau\phi')^2 - v(\partial_x\phi')^2 + g\beta^2\phi'^2\right], \quad \text{where} \quad \phi' - \phi - \frac{\pi}{\beta} \tag{16}$$

Energy spectrum and the free energy density then reads:

$$E(p) = \sqrt{p^2 + m^2}, \quad m^2 = g\beta^2 \tag{17}$$

$$F = k_B T \int \frac{dp}{2\pi} \ln\left(1 - \exp\left(-\frac{E(k)}{k_B T}\right)\right) \tag{18}$$

this allows us to see explicitly that there is a gap in the spectrum We can write down an expression for the Green's function:

$$G(\xi_1, \xi_2) = \int \frac{dk^2}{(2\pi^2)} \frac{\exp\left(i\mathbf{k}(\xi_1 - \xi_2)\right))}{k^2 + m^2} \tag{19}$$

at small distances, where $\Delta\xi = \xi_1 - \xi_2 \ll m^{-1}$ this Green's function has the same form as (9), however, when $|\Delta\xi| \gg m^{-1}$, we would see a different behavior: $G(\Delta\xi) \propto \exp(-m|\Delta\xi|)/\sqrt{|\Delta\xi|}$.

3.2 Fermionic mass gap

There is another way to see how this mass gap develops for $d = 1$, i.e. $\beta = \sqrt{4\pi}$, yielding a finite correlation length [4]. Defining the fermion operators to be [2, 5]:

$$\psi_l \propto e^{il\phi} \tag{20}$$

it can be shown via considering perturbative expansion in g that the following identity holds:

$$\int d\tau dx \left[\frac{1}{2}(\nabla\phi)^2 + g\cos(\sqrt{4\pi}\phi)\right] = \int d\tau dx \left[\bar{\psi}\gamma^\mu\partial_\mu\psi + g\bar{\psi}\psi\right] \tag{21}$$

Thus, bosonic theory perturbed by (2) with scaling dimension $d = 1$ can be interpreted as a theory of free massive fermionic field.

3.3 RG analysis

As we saw previously, Sine-Gordon model allows for two different infrared regimes: for $\beta^2 < 8\pi$ the perturbation is relevant and we obtain gapped spectrum and for $\beta > 8\pi$ the perturbation is irrelevant and the field is effectively massless.

We will now apply the renormalization group approach to study this theory near it's critical point. First of all, we introduce a cutoff to the system $|k| < \Lambda = 1/a$ and split the Brillouin zone to the "slow" components, where $|k| < \Lambda'$ and "fast" components $\Lambda' < |k| < \Lambda$:

$$\phi_\Lambda(x) = \sum_{|k|<\Lambda'} e^{ikx}\phi_k + \sum_{\Lambda'<|k|<\Lambda} e^{ikx}\phi_k \equiv \phi_{\Lambda'}(x) + h(x) \tag{22}$$

And now the partition function takes form:

$$Z_\Lambda = \int D\phi_{\Lambda'}(x)Dh(x)\ \exp\left(-S[\phi_{\Lambda'}] - S[h] - \delta S[\phi_{\Lambda'} + h]\right) \tag{23}$$

Introducing subscript $\langle ...\rangle_h$ to indicate that we averaging over field h we can write down a formal expression for an effective action after integrating out "fast" modes:

$$S_{eff}[\phi_{\Lambda'}] = S[\phi_{\Lambda'}] - \ln\langle e^{-\delta S[\phi_{\Lambda'}+h]}\rangle_h \tag{24}$$

We can deal with this expression perturbatively for small g. Expanding the logarithm we get:

$$\begin{aligned} S_{eff}[\phi_{\Lambda'}] &\approx S[\phi_{\Lambda'}] - \ln\langle 1 - \delta S[\phi_{\Lambda'} + h] + \frac{1}{2}\delta S^2[\phi_{\Lambda'} + h]\rangle_h + ... \\ &\approx S[\phi_{\Lambda'}] - \langle \delta S[\phi_{\Lambda'} + h]\rangle_h - \frac{1}{2}\left(\langle \delta S^2[\phi_{\Lambda'} + h]\rangle_h - \langle \delta S[\phi_{\Lambda'} + h]\rangle_h^2\right) + ... \end{aligned}$$

We have in the first order as was shown in [1]:

$$\begin{aligned} \langle \delta S[\phi_{\Lambda'} + h]\rangle_h &= g\int d\tau dx \langle \cos\left(\beta(\phi_{\Lambda'}(\mathbf{x}) + h(\mathbf{x}))\right)\rangle_h \\ &= \frac{g}{2}\sum_{\sigma=\pm 1}\int d\tau dx \exp(i\beta\sigma\phi_{\Lambda'})\langle \exp(i\beta\sigma h\rangle_h \end{aligned}$$

expanding this for small β:

$$\langle \exp(i\beta\sigma h\rangle_h \approx 1 - \frac{\beta^2}{2}\langle h^2\rangle_h \approx 1 - \frac{\beta^2}{4\pi}dl, \quad \text{where } dl = \frac{d\Lambda}{\Lambda} = \frac{\Lambda' - \Lambda}{\Lambda} \tag{25}$$

Inserting this into the the expression for S_{eff} and rescaling momentum $(\Lambda/\Lambda')\mathbf{k}' = (1 + dl)\mathbf{k}$ we obtain action in the old form but with renormalized value of g:

$$S_{eff}[\phi_{\Lambda'}] = S[\phi_{\Lambda'}] + g\left(1 + \left(2 - \frac{\beta^2}{4\pi}dl\right)\right)\delta S[\phi_{\Lambda'}] \tag{26}$$

And the renormalization group equation for g is:

$$g' = g\left(1 + \left(2 - \frac{\beta^2}{4\pi}dl\right)\right) \to \frac{\mathrm{d}g}{\mathrm{d}l} = \left(2 - \frac{\beta^2}{4\pi}\right)g(l) \tag{27}$$

Integrating this equation we can explicitly see why the perturbations with the value of $\beta < 0$ are relevant and perturbations with $\beta > 0$ irrelevant:

$$g(L) = \left(\frac{L}{a}\right)^{2-d}, \quad \text{where} \quad d = \frac{\beta^2}{4\pi} \tag{28}$$

as was discussed before, relevant perturbations, upon increasing the length scale, lead to the strong coupling regime, while irrelevant perturbations (d) fade out as L is increased.

Considering the second order expansion gives us renormalization of β i.e. scaling dimension d. This is not so involved as it is a rather long derivation and since it is not the main focus of this paper, we would present the final result. Technical details can be found in [1]. Second order correction to the effective action allows to derive the following RG equation for scaling dimension d:

$$\frac{\mathrm{d}d}{\mathrm{d}l} = -Ag(l)^2 d(l)^3 \tag{29}$$

where A is a nonuniversal numerical constant. These two equations known as Kosterlitz-Thouless RG equations:

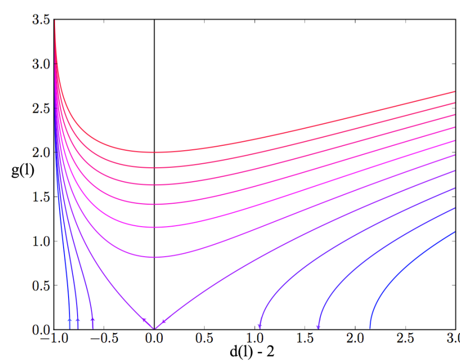

Figure 1: Phase diagram of Kosterlitz-Thouless RG equations.

$$\frac{\mathrm{d}g}{\mathrm{d}l} = \left(2 - d\right)g, \quad \frac{\mathrm{d}d}{\mathrm{d}l} = -Ag^2 d^3 \tag{30}$$

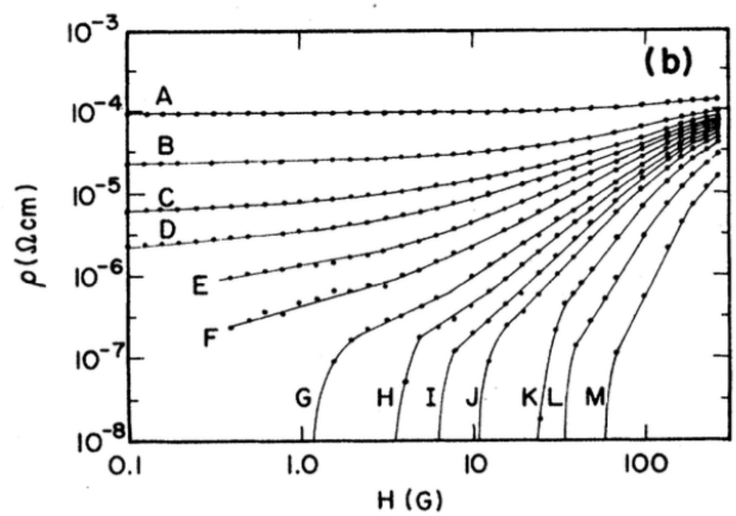

Figure 2: Magnetoresistivity data for various temperatures. T_A = 100.16K > T_B > ... > T_M = 97.88K. Graph is taken from [10].

with the phase diagram depicted on Fig. 1. This phase transition first occurred in 2D X-Y spin model. One of the most interesting features of this model is the existence of vortex-like excitations [7] which drive phase transition. In the dual 1D sine-Gordon description these excitations are described by the field ϕ. It is well known that the equations of motion for ϕ allow for topologically nontrivial solution - kink, which is analogous to 2π phase shift acquired circumventing the vortex in 2D X-Y model. BKT phase transition was probed both numerically [8] and experimentally [9] in various 2D models. In [10] electrical properties of $Tl_2Ba_2CaCu_2O_8$ thin films were examined and magnetoresistivity just above the T_c (Fig. 2) was found to be in agreement with theoretical picture (Fig. 1). But can we experimentally probe 1D sine-Gordon model *directly*? Turns out, the answer can lie in the realm of topological phases of matter which we will discuss in the next section.

4 Edge

First let us see how scalar field theory can arise on the edge of the simplest $\nu = 1/m$ FQH state. We will closely follow derivation presented in [5]. The bulk FQH state can be effectively described by the $2 + 1$D $U(1)$ Chern-Simons theory with the following action:

$$S = -\frac{m}{4\pi}\int d\tau dx dy \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \tag{31}$$

We will assume that this system covers lower half-plane of our 2D space with the boundary in x direction. Chern-Simons action is not gauge invariant but we can fix this problem introducing the boundary counter-terms for the gauge transformation $a_\mu \to a_\mu + \partial_\mu f$:

$$S_{bdy} = -\frac{m}{4\pi}\int d\tau dx f(\partial_0 a_1 - \partial_1 a_0) \tag{32}$$

restricting the gauge transformation to be zero on the boundary $f(x, y = 0, t) = 0$ and choosing the gauge condition $a_0 = 0$ and regarding the equations of motion for a_0 as a constraint, we obtain for the Chern-Simons theory the following condition: $f_{\mu\nu} = 0$ which we substitute to (31) and integrating by parts we obtain the following scalar field theory:

$$S_{edge} = -\frac{m}{4\pi}\int d\tau dx \partial_t \phi \partial_x \phi \tag{33}$$

where we introduced a scalar field via Chern-Simons gauge field: $a_\mu = \partial_\mu \phi$. This derivation can be generalized to the generic FQH states described by several gauge fields. The resulting edge theory is described by the so-called K-matrix theory:

$$S_{edge} = \frac{1}{4\pi}\int d\tau dx \left[K_{IJ}\partial_t \phi_I \partial_x \phi_J - V_{IJ}\partial_x \phi_I \partial_x \phi_J\right] \tag{34}$$

where K_{IJ} is a symmetric integer-valued matrix and V_{IJ} is renormalized charge velocity matrix.

4.1 Fractional Topological Insulator

Here we will consider a slightly more complicated model - a Fractional Topological Insulator (FTI) with filling fraction $\nu = 1/m$ where spin projection S_z is conserved. This means we have two spin species and the edge theory is described by the following Lagrangian density [12]:

$$\mathcal{L} = \frac{1}{4\pi}\sum_{\sigma,\sigma'=\uparrow,\downarrow}\left(K_{\sigma\sigma'}\partial_t \phi_\sigma \partial_x \phi_{\sigma'} - V_{\sigma\sigma'}\partial_x \phi_\sigma \partial_x \phi_{\sigma'}\right) \tag{35}$$

with $K = m\sigma_z$. Introducing new field variables $\varphi = \frac{m}{2}(\phi_\downarrow + \phi_\uparrow)$ and $\theta = \frac{1}{2}(\phi_\downarrow - \phi_\uparrow)$ and performing Legendre transformation we can write down Hamiltonian of the edge theory in a rather concise form:

$$H_{edge} = \int dx \frac{u}{2\pi} \left[mg(\partial_x \theta)^2 + (mg)^{-1}(\partial_x \varphi)^2 \right] \tag{36}$$

where $g = 1$ if $V_{\uparrow\downarrow} = 0$ and $g > 1$ if $V_{\uparrow\downarrow} < 0$ ($g < 1$ if $V_{\uparrow\downarrow} > 0$). Fermionic creation operators in this theory are given by $\psi_\sigma^\dagger = \frac{1}{\sqrt{2\pi\alpha}} e^{iK_{\sigma\sigma'}\phi_{\sigma'}}$. Coupling this edge to superconductor effectively introduces the following term to the edge theory:

$$\mathcal{H}_{SC} = \Delta e^{i\chi} \psi_\uparrow^\dagger \psi_\downarrow^\dagger + h.c. = \frac{\Delta}{\pi\alpha} \cos(2m\theta + \chi) \tag{37}$$

where Δ is the induced superconducting gap and χ is superconducting phase. As we saw previously in this paper the behavior of Δ under RG is given by [12]:

$$\frac{\mathrm{d}\Delta(l)}{\mathrm{d}l} = \left(2 - \frac{m}{g}\right) \Delta(l) \tag{38}$$

Instead of coupling this theory to superconductor we can turn on magnetic field which would introduce coupling a different coupling, instead of $\psi_\uparrow^\dagger \psi_\downarrow^\dagger$ we would have a series of n electron backscattering terms $\psi_\uparrow^\dagger \psi_\downarrow \propto \exp(inm(\phi_\uparrow + \phi_\downarrow))$ leading to the following perturbation:

$$H_M = \sum_{n=1}^{\infty} \int_0^{L_M} dx a_n [F_n e^{i(2n\varphi)} + h.c.] \tag{39}$$

It was shown in [13] that if the sector where the magnetic field is present is long enough $L_M \gg L_T$, where $L_T = \frac{u}{k_B T}$, the system would be governed by a sine-Gordon Hamiltonian leading to a Kosterlitz-Thouless RG equations:

$$\frac{\mathrm{d}r_Z}{\mathrm{d}l} = (2 - mg) r_Z \tag{40}$$

$$\frac{\mathrm{d}g}{\mathrm{d}l} = -A r_Z^2 g^2 \tag{41}$$

where A is nonuniversal constant, $r_z \propto a_1/E_c$, and E_c is a high energy cutoff. Physically phase transition diagram allows to separate conducting and insulating phases as depicted in Fig. 3. This result means that we probably should be able to detect FTI phases measuring conductance on the edge of a 2D crystal varying temperature and magnetic field.

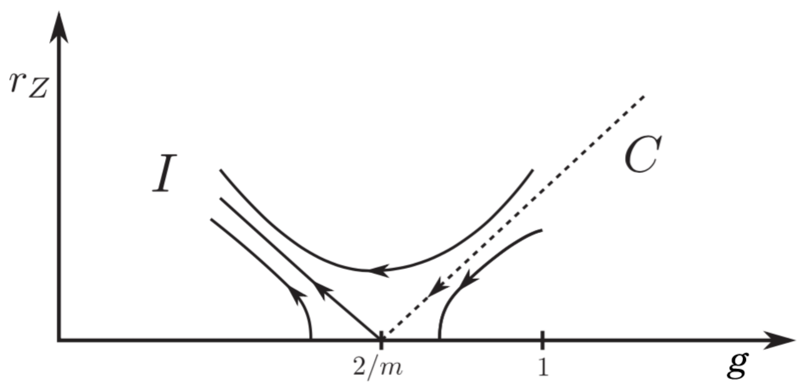

Figure 3: Kosterlitz-Thouless phase diagram for FTI in magnetic field. Dashed line separates conducting and insulating phases. Figure is taken from [13].

5 Conclusion

In this term paper we reviewed 1D sine-Gordon model and presented derivation of renormalization group flows and showed that they represent RG flows of 2D X-Y spin model which was probed in several experiments as we mentioned briefly. We also discussed how these results can be applied to the edge theory of Fractional Topological Insulators and presented several recent results suggesting that topological phases of matter can be detected by experimental study of their edge properties, which must exhibit conducting and insulating edge phases in accord with KT phase diagrams.

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