Spontaneous Symmetry Breaking and Goldstone Bosons – Applications in a Broad Range of Physical Systems

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Abstract: The concept of Goldstone bosons that arises as consequence of the breaking of continuous symmetry was born in the high energy physics and found interesting applications in other fields of physics such as solid state. This is a review assay of the reference articles where a summary of basic theory and some interesting applications are outlined.

1. MOTIVATION

We can argue that if there is to be a general theory of physics, it should be capable of accounting for phenomena occurring at different energy scales: low(solid state physics), middle(nuclear physics) and high(particle physics). We know that quantum mechanics is the basis, however the techniques that are specific to each field have been on the divergent paths from others that made difficult to attain an integrate picture of matter as a whole.

However, there is a language that is applicable to all of the energy ranges: that of the field theory either classical or quantum mechanical, that exploits the properties of symmetry and the simplification of the problem by low-energy expansion. It is in this context that the theory of Goldstone bosons acquires its powerful application.

In short, Goldstone bosons are massless degrees of freedom that appear as consequence of the breaking of the continuous symmetry of the ground state. These modes or 'particles' become decoupled in the low energy limit and exhibit the following additional properties:

1). Goldstone particles can be represented by fields that are rotational scalars, hence these particles are spinless bosons.

2). There is precisely one Goldstone particle for each symmetry generator that is broken.

In this essay, I will try to summarize the underlying basic theoretical concepts and some physical examples; starting with pion-pion scattering in the high energy physics where this whole approach first found its application and experimental verification and then discuss about some relevant and on-going topics of the solid state physics.

2. THEORY

2.1. BACKGROUND

We can simplify a great deal of problem by considering the lower energy (or temperature) limit of a physical system. There are fewer degrees of freedom and the interactions between the states become less important in many cases. A generally accepted method of studying low energy systems is to formulate 'effective lagrangians'. The formulation of the problem in the language of the field theory presupposed the identification of the relevant low-energy degrees of freedom and symmetries. It seem like the choice of the variables is itself a critical part of the problem. To supplement the effectiveness of this approach there are power laws that allows simple identification of the interaction terms.

First of all, two important theorems should be remarked, as this whole approach is based on them:

+The first one is Noether's theorem: This theorem states that there is a conserved current j^{μ} whenever the action is globally symmetric. Introducing the language of the field theory the action is $S = \int d^4 x L(\phi, \partial_{\mu}\phi)$, where $\phi(x)$ denotes the fields relevant to the problem. Let's say that the fields transform according to $\delta\phi = \xi_a(\phi)\omega^a$, then the invariance of S implies the transformation of the lagrangian density at most by a total derivative: $\delta L = \partial_{\mu}(\omega^a V_a^{\mu})$ for some quantity $V_a^{\mu}(\phi)$. Noether's theorem states that if we define $j_a^{\mu} \equiv -\frac{\partial L}{\partial(\partial_{\mu}\phi)}\xi_a + V_a^{\mu}$, this quantity satisfies the relation: $\partial_{\mu} j_a^{\mu} = 0$. The last relation is valid for relativistic as well as non-relativistic systems. It is easy to notice that this is analogous to the usual continuity equation. Thus we can define as 'charge' $Q_a(t) \equiv \int d\vec{x} j_a^0(x)$. The existence of such a 'current' that obeys the conservation law gives rise to the following theorem.

+Goldstone's theorem: When a symmetry is broken (by a system's ground state), weakly coupled states known as Goldstone bosons appear. If this state is denoted by $|G>$, then the matrix element of the density of charge $<\Omega| j^0 |G>$ cannot vanish; where $|\Omega>$ is the ground state of the system. The proof is based on the property that the charge as defined by Noether's theorem is the generator the symmetry transformation ($\delta\phi = i[Q, \phi(x)]$) and the fact that the field $\phi(x)$ must have a nonzero expectation value in the ground state: $<\Omega| \phi(x) |\Omega> \neq 0$.
The important consequences of this theorem are:

i). The Goldstone boson must be 'gapless', that is when the momentum vector goes to zero the energy must also go to the zero limit. We can easily see that this is equivalent to the masslessness of the Goldstone particle in the relativistic systems: $E(p) = \sqrt{p^2 + m^2}$.

ii). In the case of the exact symmetry, the Goldstone bosons become completely decoupled of all the interactions in the limit that their momentum goes to zero.

In many real cases, we don't have 'an exact symmetry'. However, we can treat the system as basically symmetric with some perturbations that account for the violations of the above-mentioned properties. The approximate symmetry particles are known as pseudo-Goldstone bosons, which can have light mass and weak interactions.

In general, the lagrangian does not show explicitly the property that the couplings become weaker for the limit of vanishing momentum. This is in fact hidden from the simple identification and becomes apparent once the scattering amplitude is calculated. We can adopt two postures in such a case: to keep the lagrangian as original which displays the renormalizability or to make a change of variables that manifest in the lagrangian the Goldstone modes; the latter at the expense of losing the renormalizability. In the context of the current discussion, this last choice is the preferred one. We have to remember that the symmetry transformations are mathematical in nature and do not depend on the details of the actual physical model, thus the generality of the method.

The symmetry operations that play role in Noether's theorem and hence also in Goldstone's theorem belong to the so called continuous symmetry group; that is, the group elements can be parametrized by a continuous parameter (as the angle for the operation of rotation) as opposed to a discrete label. These groups obey Lie algebra and, not surprisingly, are called Lie groups. The properties of Lie algebra are well known mathematically of which we can denote that:

i). It allows representation as finite-dimensional unitary matrices.

ii).There exist the so-called generators T_a, which are finite-dimensional and hermitian matrices. These matrices obey a defining commutation relation. A symmetry operation can be written as: $g = \exp[i\alpha^a T_a]$.

2.2. GENERAL APPROACH

Let λ be a parameter that gives the strength of the interaction potential, we can take a semi-classical approach for the limit $\lambda << 1$. In this context we find the minimum (v) of the scalar potential and suppose that the low-degrees of freedom are the small harmonic oscillations of the field about the minimum of the scalar potential. If we define the real and imaginary parts of the field fluctuations as $R \equiv \sqrt{2}\,\mathrm{Re}(\phi - v)$ and $I \equiv \sqrt{2}\,\mathrm{Im}(\phi)$, we can expand the scalar potential in powers of *R* and *I*. From this expansion it becomes apparent the corresponding Feynmann diagrams of different order and contributions to the lagrangian.

3. EXAMPLES

3.1. APPLICATIONS IN HIGH ENERGY (PIONS)

Let's remember that modern description of the strong interactions is based on the theory of interaction of spin-half quarks and spin-one gluons. This is the so called Quantum Chromodynamics(QCD). This theory is described by the lagrangian density:

$$L_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a - \sum_n \bar{q}_n (\gamma^\mu D_\mu + m_{q_n}) q_n$$

where G^a_μ is the field strength tensor for the gluon fields and *a =1 ~ 8* labels the generators of the gauge symmetry group of the theory. The quarks are described by q_n where *n = 1~6*; that is there are six different kinds(flavors) of quarks, and γ^μ are the Gell-Mann matrices.

It is this kind of lagrangians that give rise to the bound states (hadrons) such as pions (π) that are composed of a pair of quarks. In the limit of vanishing masses for the constituent quarks we have the exact symmetry of QCD known as 'chiral symmetry'; so called because it treats left handed fermions (quarks) differently from the right handed fermions.

As the chiral symmetry is only approximate, we have a better agreement of the theory at light mass cases; that is when the building quark masses are much lighter than the scale of the strong interactions. This is precisely the case of the pions ($\pi^{\pm}$and π^{0}). These modes correspond to the Goldstone bosons for the symmetry breaking of $SU_L(2)\times SU_R(2)\rightarrow SU_I(2)$.

Starting from a generic lagrangian density that is invariant upon symmetry operation $g \in G$ and assuming canonical normalization of the pion fields, it is possible to obtain explicitly the pion-pion interaction lagrangian (density) as well as pion-nucleon interaction lagrangian:

$$L_{\pi\pi} = -\frac{1}{2}\partial_\mu \vec{\pi}\partial^\mu \vec{\pi} - \frac{1}{2F^2}(\vec{\pi}\cdot\partial_\mu\vec{\pi})(\vec{\pi}\cdot\partial^\mu\vec{\pi}) + O(\pi^6)$$

$$L_{\pi N} = -\overline{N}(\gamma^\mu\partial_\mu + m_N)N - \frac{ig}{2F}(\overline{N}\gamma^\mu\gamma^5\vec{\tau}N)\cdot\partial_\mu\vec{\pi} - \frac{i}{2F^2}(\overline{N}\gamma^\mu\vec{\tau}N)\cdot(\vec{\pi}\times\partial_\mu\vec{\pi}) + \cdots$$

Notice that the quantities such as F and g are to be determined from the experimental input. Now, from these lagrangians it is possible to construct the conserved currents of Noether. The current that corresponds to the isospin symmetry is:

$$\vec{j}_I^\mu = \vec{j}_L^\mu + \vec{j}_R^\mu = \frac{i}{2}\overline{q}\gamma^\mu\vec{\tau}q$$

and the one that corresponds to the broken symmetry is:

$$\vec{j}_A^\mu = \vec{j}_L^\mu - \vec{j}_R^\mu = \frac{i}{2}\overline{q}\gamma_5\vec{\tau}q$$

What is useful about these quantities is that we can compare with the experiment when we consider the process of the transition of d quarks into u quarks. The most commonly known of such processes are the beta decay, free-neutron decay and $\pi^{\pm}$ decay. In all these cases, the decay rate can be measured and compared with the theory. The decay theory is described by the weak interaction lagrangian term built from the currents.

$$L_{weak} = \frac{G_F\cos\theta_C}{\sqrt{2}}\overline{u}\gamma^\nu(1+\gamma_5)d\overline{v}_l\gamma_\nu(1+\gamma_5)^l + \cdots$$

where G_F and θ_C are measurable quantities. To compute the decay rate for the process $\pi^+ \rightarrow \mu^+\nu_\mu$, we have to calculate the matrix element $<\mu^+,\nu_\mu|L_{weak}|\pi^+>$. From the experimental measurement of this process it is possible to infer that F = 92 MeV. Similarly, we can also estimate g experimentally, which is approximately 1.26. From now and on, the theory acquires predictive powers.

A case in which we can compare the theory with experiment is the that of pion-pion scattering($\pi_a \pi_b \rightarrow \pi_c \pi_d$), where the scattering matrix(S) element is:

$$S(\pi_a \pi_b \rightarrow \pi_c \pi_d) = \frac{i\delta^4(p_a + p_b - p_c - p_d)}{(2\pi)^2 \sqrt{p_a^0 p_b^0 p_c^0 p_d^0}} A_{ab,cd}$$

with

$$A_{ab,cd} = A^{(0)} \frac{1}{3} \delta_{ab} \delta_{cd} + A^{(1)} \frac{1}{2} (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}) + A^{(2)} \left[\frac{1}{2} (\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) - \frac{1}{3} \delta_{ab} \delta_{cd} \right]$$

The amplitudes $A_l^{(I)}$ can be expanded as:

$$A_l^{(I)} = \left(\frac{q^2}{m_\pi^2} \right)^l \left(a_l^I + b_l^I \frac{q^2}{m_\pi^2} + \cdots \right)$$

The quantities: a_l^I and b_l^I are the ones that are observed experimentally and we can easily corroborate the validity of the predictions.

Parameter	**Leading Order**	**Next Order**	**Experiment**
a_0^0	0.16	0.20	0.26(5)
b_0^0	0.18	0.26	0.25(3)
a_1^1	0.030	0.036	0.038(2)
a_0^2	-0.044	-0.041	-0.028(12)
b_0^2	-0.089	-0.070	-0.082(8)

Although we had to determine experimentally the decay constant F, the low-energy effective lagrangian method is capable of predicting many observables. What is important is that we can introduce more terms in the calculation and fine-tune the theoretical calculations. This conditional predictability arises as a result of the symmetries and the restrictions on the interactions at low-energy expansion.

3.2. APPLICATIONS IN SOLID STATE (MAGNONS)

It is a known fact that the magnetic systems (ferromagnetic or antiferromagnetic) exhibit the behavior of phase transition of the bulk order parameter at low temperature. In the case of the ferromagnets, this order parameter can be taken as the total magnetization. As this order parameter breaks spontaneously the rotational symmetry (G=O(3) $\rightarrow$ H=O(2)), Goldstone bosons are expected. The corresponding Goldstone modes are referred as spin waves or magnons. The physical picture is the twisting of the local spin orientation while the wave passes through the crystal like plane waves

In contrast to the relativistic case (as the previous example), the exact form of the dispersion relation at large wavelengths can not be specified by Goldstone's theorem nor the number of Goldstone 'particles'. These properties are not determined by symmetry considerations alone: they depend on the specifics of the (non-relativistic) system. Only the number of real Goldstone fields turns out to be universal, equal to the dimension of the coset G/H.

Even if we have the same symmetry breaking pattern, the ferromagnetic and antiferromatic systems differ in their qualitative description of the dispersion relation which is in agreement with the statement of the last paragraph. Moreover, the numbers of independent magnon states differ for the two cases: one for a ferromagnet and two for an antiferromangnet.

As usual, the theory of magnetic systems starts by writing the effective lagrangians:

$$L_{eff}^{F} = \Sigma \frac{\partial_0 U^1 U^2 - \partial_0 U^2 U^1}{1+U^3} + \Sigma f_0^i U^i - \frac{1}{2} F^2 D_r U^i D_r U^i$$
$$L_{eff}^{AF} = \frac{1}{2} F_1^2 D_0 U^i D_0 U^i - \frac{1}{2} F_2^2 D_r U^i D_r U^i + \Sigma_S \mu h^i U^i$$

where: U^i = magnon fields, $f_0^i = \mu H \delta_3^i$, Σ= order parameter and F^i= low energy coupling constant.

The corresponding equations of motion (Landau-Lifshitz) is of first order in time and second order in space for the ferromagnetic lagrangian and second order both in time and space for the anti-ferromagnetic case.

It is possible to obtain the expression for the order parameter for both cases as expansion in terms of the temperature:

+For the anti-ferromagnetic case ($F = F_1 = F_2$):

$$\Sigma_S(T) = \Sigma_S \left\{ 1 - \frac{N-1}{24} \frac{T^2}{F^2} - \frac{(N-1)(N-3)}{1152} \frac{T^4}{F^4} - \frac{(N-1)(N-2)}{1728} \frac{T^6}{F^6} \ln(\frac{T_\Sigma}{T}) + O(T^8) \right\}$$

each one of these power terms can be interpreted as the loop graphs(invoking again the methods commonly used in the high energy physics).

+For the ferromagnetic case (order parameter = spontaneous magnetization)

$$\Sigma(T) = \Sigma\left\{1 - \alpha_0 T^{\frac{3}{2}} - \alpha_1 T^{\frac{5}{2}} - \alpha_2 T^{\frac{7}{2}} - \alpha_3 T^4 + O(T^{\frac{9}{2}})\right\}$$

where half integer powers can be interpreted as corresponding to the non-interacting magnons.

It is noteworthy the difference of time-reversal symmetry (T) for both systems. The time-reversal invariance is the one that changes the sign of each spin. Although this would also reverse the order parameter, the combination of T with another broken symmetry (S = shift of the whole lattice by a single lattice site in the antiferromagnetic system) would be a symmetry operation(TS) of the order parameter. In the case of ferromagnetic system it is not possible to find the broken symmetry with which to combine the time reversal invariance T and to preserve the total order parameter (magnetization). The important difference that arises is that in latter case, the effective lagrangian contains terms that violate T invariance. Also, the absence of the derivatives of the Goldstone boson fields is the consequence of violating T invariance.

Just to mention briefly about the experimental measurements; scattering of the spin-polarized neutrons is capable of detecting magnetic structures. Inelastic neutron scattering can be used to detect magnons in magnetic systems, the same as it is used to measure phonons. There is an ample source of experimental data for this method. As far as the theory is concerned, we can identify terms of lagrangian that couple with the external magnetic field and calculate the scattering cross section.

4. PERSPECTIVES AND DISCUSSIONS

In this section theories regarding superconductivity are discussed. These theories are fairly new and still needing experimental corroboration.

The first one is a bold proposal of the existence of SO(5) symmetry invariance in the cuprates while exhibit high temperature superconductivity. This is motivated by the experimental fact that relates superconductors with antiferromagnet: that by altering ‘doping’ in small amount it is possible to convert high Tc superconductors into antiferromagnets. The proposal is about relating these two phases by an approximate SO(5) symmetry. If we consider the order parameter space, we know that the antiferromagnetic(AF) systems have SO(3) symmetry while the superconductors(SC) have SO(2) symmetry. We can suppose that the system (cuprate) is invariant upon 5x5 rotations in the order parameter space. The symmetry operation would be in a block diagonal form of SO(2)xSO(3).

Although this model is not widely accepted yet, there are some features that are taken as valid:

i). The breaking of SO(5) symmetry, if valid, must generate 4 goldstone bosons(counting also the pseudo-bosons).

ii). The low energy properties of the Goldstone particles are independent of the details of the system and obey primarily to the symmetry breaking pattern.

This theory predicts among other things the existence of spin-triplet pseudo-Goldstone state in the SC phase, and an electrically-charged state in the AF phase. The SO(5) invariance can determine unambiguously the relative properties of these states such as the gap, phase velocity, etc.

Furthermore, the spin-triplet state is believed to have been observed with a gap of 41 meV. If true, this would pave the path toward the experimental search for the measurement of Goldstone bosons in AF phase.

Finally, there is another interesting paper by X.Z. Yan(reference) in which pairing fluctuation effects in quasi 2-D superconductors are studied; we can regard high Tc superconductors as quasi-2D systems. In this work, it is proposed that the fluctuation effects might be important in accounting for the superconductivity. This is supposed to be an improvement over the perturbation methods: the Goldstone modes are taken as the excitation of the pairs that are the predominant fluctuations. The existence of the Goldstone modes is justified by the breaking of symmetry by coherent pairing. The quantity known as pseudo-gap parameter can be calculated by this theory.

In conclusion, all these topics are hot out of the press waiting for further refinement and experimentation.

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