Q-Balls

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Abstract

Q-balls are soliton-like solutions that exist in certain types of scalar field theories and they appear naturally in Sypersymmetric extensions of Standard Model (SSM). In the first section of the paper the basic formalism demonstrating the existence of Q-balls is outlined. Second section talks about emergence of Q-Balls in SSM. Third section talks how such objects can be formed in the early univesre. In the final section some observational aspects of Q-Ball detection are discussed. Appendix proves a useful general theorem which is required for establishing existence of Q-balls.

I Q-ball formalism

Q-balls are a class of extended objects that appear in a large family of field theories. These objects carry charge Q with resepct to an internal symmetry and are spherically symmetric. Q is a measure of particle number as well as the volume of the object and the object as a whole can be considered as a particle with charge Q. In this section, a justification for existence of Q balls would be given along with the required conditions.[1] Consider a simple SO(2) invariant theory of two real scalar fields given by lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - U(\phi) \tag{1}$$

where $\phi = \sqrt{\phi_1^2 + \phi_2^2}$ and the SO(2) symmetry is given by

$$\phi_1 \rightarrow \phi_1 \cos\alpha - \phi_2 \sin\alpha, \tag{2}$$
$$\phi_2 \rightarrow \phi_1 \sin\alpha + \phi_2 \cos\alpha \tag{3}$$

Associated conserved current and conserved charge are given by

$$j_\mu = \phi_1 \partial_\mu \phi_2 - \phi_2 \partial \phi_1 \quad (4)$$
$$Q = \int d^3\mathbf{x} j_0 \quad (5)$$

Although the formalism in this section would be devloped for a field theory lagrangian given in equation (1), it can be extended to theories with more complicated internal group of symmetry. Section II would discuss how this formalism can be applied to Minimal Supersymmetric Standard Model (MSSM). Now, one can adjust the potential $U(\phi)$ such that $U(0) = 0$. $\phi = 0$ is the absolute minima for the potential and mass of the particle representing field ϕ carrying charge $Q = \pm 1$ is given by $\mu^2 = U''(0)$. The required condition for existence Q-balls is:

$$min[2U/\phi^2] \equiv 2U_0/\phi_0^2 < \mu^2 \quad (6)$$

When this condition is satisfied, for a large enough value of Q these objects are absolutely stable (being minima of energy for a given value of conserved charge Q) and not just stable under small deformations. The only relevant renormalizable interaction allowed by Quantum field theory is: $U(\phi) = \frac{1}{2}\mu^2\phi^2 + \lambda\phi^4$. Such a function can never satisfy equation (6), but one need not have U to be a fundamental interaction and it can be some effective interaction. So as long as one is concerned with leading order phenomenon, the loop effects that distinguish the fundamental and effective interactions would not be relevant. The Q-ball solution is given as (justification of this type of solution which rotates in the internal space is given in next section, eqn.(15))

$$\phi_1 = \phi(r)\cos\omega t$$
$$\phi_2 = \phi(r)\sin\omega t$$

with $\phi(r)$ a monotonically decreasing function of r. To prove existence of Q balls, consider a approximate form for a Q-ball: ϕ is constant in some volume V and ϕ is zero outside. The energy is given by

$$E = \int d^3\mathbf{x} \left[\frac{1}{2}\dot{\phi_1}^2 + \frac{1}{2}\dot{\phi_2}^2 \frac{1}{2}(\nabla\phi_1)^2 + \frac{1}{2}(\nabla\phi_2)^2 + U(\phi)\right] \quad (7)$$

In the approximation of a constant ϕ inside volume V and zero elsewhere, the energy becomes:

$$E = \frac{1}{2}\omega^2\phi^2 V + U(\phi)V \quad (8)$$

While the charge Q from (5) is given by:

$$Q = \omega\phi^2 V \tag{9}$$

Thus energy can be written as:

$$E = \frac{1}{2}\frac{Q^2}{\phi^2 V} + U(\phi)V \tag{10}$$

which has a minima, as a function of V, at $V = Q/\sqrt{2U/\phi^2}$ and energy for this configuration becomes

$$E = Q\sqrt{2U/\phi^2} \tag{11}$$

Thus energy per unit charge is given by $\sqrt{2U/\phi^2}$ and from (6) its minimum value is less than μ, the particle mass. This proves that the Q ball has less energy than a set of free particles carrying total charge Q and hence is a stable state. Although this proves that the energy of Q ball state is less than free particles state, it doesn't prove that a spherically symmetric state is the one that is energetically the most favoured one. In the appendix an powerful theorem would be proved which states that a spherically rearranged configuration has less total energy than the original configuration.[2]

II Q-balls and MSSM

Having established basic ingredients going into formation of Q-balls in the last section, in this section, we would see how these can appear in particle physics models. Supersymmetric generalizations of standard model (SSM) admit Q-balls which may have non-zero baryon number and lepton numbers. SSM involves a scalar potential that depends on the model. Before looking at a toy model of Minimal SSM (MSSM), analysis of last section would be generalised for a case that involves several complex scalar fields with different charges.[3] Consider a potential of the form, $U(\phi) \equiv U(\phi_1, ..., \phi_n)$ which has a global minima at $\phi = 0$. $U(\phi)$ and has an unbroken global U(1) symmetry and fields ϕ_i have charges q_i with respect to the symmetry. The charge Q is defined to be (generalising eqn. (5))

$$Q = \sum_k q_k \frac{1}{2i}\int (\phi_k^* \partial_t \phi_k - \phi_k \partial_t \phi_k^*) d^3\mathbf{x} \tag{12}$$

The energy is given by (generalising eqn. (7))

$$E = \int d^3\mathbf{x}\left[\frac{1}{2}\sum_k |\dot{\phi_k}|^2 \frac{1}{2}\sum_k |\nabla\phi_k|^2 + U(\phi)\right] \tag{13}$$

To find extremum of the energy for a fixed charge Q one introduces lagrange multiplie ω

$$\begin{aligned}\mathcal{E}_\omega &= E + \omega\left[Q - \sum_k q_k \frac{1}{2i}\int(\phi_k^*\partial_t\phi_k - \phi_k\partial_t\phi_k^*)d^3\mathbf{x}\right] \\ &= \int d^3\mathbf{x}\frac{1}{2}\sum_k |\partial_t\phi_k - i\omega q_k\phi_k|^2 + \int d^3\mathbf{x}\left[\frac{1}{2}\sum_k |\nabla\phi_k|^2 + \hat{U}_\omega(\phi)\right] + \omega Q\end{aligned}$$

where $\hat{U(\phi)}_\omega$ is given by

$$\hat{U(\phi)}_\omega = U(\phi) - \frac{1}{2}\omega^2 \sum_k q_k^2|\phi_k|^2 \tag{14}$$

To minimize the $\mathcal{E}_\omega$ one must choose

$$\phi_k(\mathbf{x}, t) = e^{iq_k\omega t}\phi_k(\mathbf{x}) \tag{15}$$

Thus Q-balls are built of fields that rotate in the internal space with velocities proportional to charges associated with the fields. Charge, Q and energy $\mathcal{E}_\omega$ is given by

$$Q = \omega \sum_k q_k \int \phi_k^2(\mathbf{x})d^3\mathbf{x} \tag{16}$$

$$\mathcal{E}_\omega = \int d^3\mathbf{x}\left[\frac{1}{2}\sum_k |\nabla\phi_k(\mathbf{x})|^2 + \hat{U}_\omega(\phi(\mathbf{x}))\right] + \omega Q \tag{17}$$

As proved in the appendix and in first section the Q balls exist if μ^2 as defined below attends its minima at $\phi \neq 0$.

$$\mu^2(\phi) = \frac{2U(\phi)}{\sum_k q_k\phi_k^2} \tag{18}$$

Having generalised the Q-ball formalism to multiple filds, we can now look at a partcular model for super symmetry.[3] A generic feature of the SSM is presence of interaction of the form $H\Phi\phi$, where H is higgs field, Φ is a left-handed squark ($\tilde{Q}_L$) or slepton ($\tilde{L}_L$), and ϕ is corresponding right-handes singlet ($\tilde{q}_R$ or $\tilde{l}_R$). To see how Q-balls (where Q can be lepton number, baryon number or electric charge) can arise in MSSM, consider following toy model of interactions:

$$U = m_H^2|H|^2 + m_L|\tilde{L}_L|^2 + m_l^2|\tilde{l}_R|^2 - yA(H\tilde{L}_L^*\tilde{l}_R + c.c.) + y^2(|H^2\tilde{L}_L|^2 + |H^2\tilde{l}_R|^2 + |\tilde{l}_R^2\tilde{L}_L|^2 + V_D \tag{19}$$

where V_D is given by

$$V_D = g_1^2/8[|H|^2 + |\tilde{L}_L|^2 - 2|\tilde{l}_R|^2]^2 \tag{20}$$

One can see that the potential is invariant under the global lepton number symmetry ($\tilde{L}_L \rightarrow exp(i\theta)\tilde{L}_L$ and $\tilde{l}_R \rightarrow exp(i\theta)\tilde{l}_R$. The fields $\tilde{L}_L$ and $\tilde{l}_R$ both have unit charge under this symmetry and hence the μ^2 can be written as follows: Define

$$\begin{aligned} H &= F\sin\xi \\ \tilde{L}_L &= F\cos\xi\sin\theta \\ \tilde{l}_R &= F\cos\xi\cos\theta \end{aligned}$$

$$\mu^2(F) = \frac{2U}{|\tilde{L}_L|^2 + |\tilde{l}_R|^2} = \frac{1}{\cos^2\xi}\left[\gamma_2(m_i^2, \xi, \theta) - yA\gamma_3(\xi, \theta)F + \gamma_4(\xi, \theta)F^2\right] \tag{21}$$

where γ_2 and γ_4 and non-negative functions of masses and mixing angles and $\gamma_3 = \cos^2\xi\sin\xi\sin(2\theta)$. The requirement for existence of L-balls would be occurance of minima of μ^2 at $F \neq 0$ which is possible if $yA \neq 0$. Thus lepton balls would exist as long as the trinilinear coupling constant y is not zero. A similar treatment can be applied to prove the existence of baryon-balls.

III Q-ball cosmology

Having established existence of Q-balls in SSM, we would look how these can be produced effectively in the early universe. Dark matter is one of the most intriguing mysteries of modern cosmology and Q-balls can be considered as candidates to form certain fraction of Dark-matter. A natural question that one needs to address is whether Q-balls produced can survive till present epoch. Solitogenesis (i.e. Q-ball formation through charge-accretion) and collision induced mergers can produce Q-balls, but they are not large enough to survive till present epoch [4]. Q-balls can be produced at the end of inflation through a process of condensate disintegration. Homogeneous scalar condensates arise at the end of inflationary cosmology and are associated with Afflek-Dine baryogenesis. A combination of supersymmetric fields can acquire some large value along some flat direction (one that grows slower than square of the field value) in the potential. At large VEV, high-energy physics can violate baryon number which along with CP violating interactions can result in the condensate acquiring a net baryon number. The resultant object can be considered to be made up of Q-matter (or B-matter). The primordial fluctuations in the condensate can grow exponentially and become non-linear in an expanding Q-matter dominated region of the universe. Thus the condensate can fragment into small Q-balls of size that corresponds to the scale at which the fluctuations first enter non-linear regime. We shall first see how condensate modes can grow non-linear [4] and then estmate the typical size of emerging Q-balls [5]. Writing the scalar field as $\phi = Re^{i\Omega}$ and working in standard FRW cosmology one gets, from Einstein equations, the equation of motion for the R and Ω fields as[4]

$$\ddot{\Omega} + 3H\dot{\Omega} - \frac{1}{a^(t)}\nabla^2\Omega + \frac{2\dot{R}}{R}\Omega - \frac{2}{a^2(t)R}(\partial_i\Omega)(\partial^i R) = 0 \quad (22)$$

$$\ddot{R} + 3H\dot{R} - \frac{1}{a^(t)}\nabla^2 R + \dot{\Omega}^2 R - \frac{1}{a^2(t)}(\partial_i\Omega)^2 R + \frac{\partial U}{\partial R} = 0 \quad (23)$$

where H is hubble constant. From the equations of motion one can get the equations for small perturbations $\delta\Omega$ and δR as[4]:

$$\delta\ddot{\Omega} + 3H(\delta\dot{\Omega}) - \frac{1}{a^(t)}\nabla^2(\delta\Omega) + \frac{2\dot{R}}{R}(\delta\Omega) - \frac{2\Omega}{R}(\delta\dot{R}) - \frac{2\dot{R}\dot{\Omega}}{R^2}\delta R = 0 \quad (24)$$

$$\delta\ddot{R} + 3H(\delta\dot{R}) - \frac{1}{a^2(t)}\nabla^2(\delta R) - 2R\dot{\Omega}(\delta\dot{\Omega}) - U''\delta R - \dot{\Omega}^2\delta R = 0 \quad (25)$$

To examine the stability of the homogeneous condensate solution one can consider the perturbation $\delta\Omega, \delta R \propto e^{S(t) - i\mathbf{kx}}$ against the background $\phi(\mathbf{x}, t) = \phi(t) \equiv R(t)e^{\Omega t}$. the growing modes for the equation motion for perturbations would be those having $\alpha \equiv dS/dt > 0$. One thus obtains the dispertion relation as[4]:

$$\left[\alpha^2 + 3H\alpha + \frac{k^2}{a^2} + \frac{2\dot{R}}{R}\alpha\right] \times \left[\alpha^2 + 3H\alpha + \frac{k^2}{a^2} - \dot{\Omega}^2 + U''(R)\right] + 4\dot{\Omega}^2\left[\alpha - \frac{\dot{R}}{R}\right]\alpha = 0 \quad (26)$$

If $(\dot{\Omega}^2 - U''(R)) > 0$, there is a range of growing modes $\alpha(k), 0 < k < k_{max}$ with $k_{max}(t) = a(t)\sqrt{\dot{\Omega}^2 - U''(R)}$. If $k_{max}(t)$ increase with time or stays constant, the modes in the window would undergo growth and would eventually lead to the disintegrations of the condensate into Q-balls. If, on the other hand, $k_{max}(t)$ decreases with time, there would only finite amount of time available for each mode to grow and the disintegration may not complete. To see if the Q-balls can be formed in a given model one need to look at a mode which maximizes the value

$$S_k = \int \alpha(k,t)dt \tag{27}$$

While the condensate disintegrates, the thermal fluctuations can possibly erode the coherent state, but this is prevented if the VEV is larger than gT where g is coupling constant for decaying interaction.

Now, consider following potential that can arise in supersymmetry breaking

$$U(\phi) = m_s^4 \ln\left(1 + \frac{\phi^\dagger\phi}{m_s^2}\right) - cH^2\phi^\dagger\phi + \frac{\lambda^2}{m_p^2}(\phi^\dagger\phi)^3 \tag{28}$$

For small $|\phi| \sim m_s$, the first term in potential dominates and so $U(\phi) \approx m_s^2\phi^\dagger\phi$. Equation for motion for R can be solved giving: $R \sim 1/t$ which gives $\dot{\Omega} \sim U''$ which implies $k_{max} \sim a$. Thus the window of growing modes $0 < k < k_{max}$, although small doesn't shrink and the condensation disintegration into Q balls can be achieved.

After seeing that the Q-balls can indeed form in the early universe by disintegration of scalar fields condensate, we can try to estimate their size[5]. The pressure for a scalar field evolution can be defined to be

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{29}$$

For a slowly moving field, or for a field trapped in a false vacuum ($\dot{\phi}^2 \approx 0$), this can be negative and can give rise to exponential expansion.

$$\begin{aligned} p &= (K/2)\rho \\ \delta\ddot{\phi}_k &= \frac{-Kk^2}{2}\delta\phi \\ \delta\phi_k(t) &= \delta\phi_k(0)exp\left(\left(\frac{|K|k^2}{1/2}\right)^2 t\right) \end{aligned}$$

The time at which the perturbation of scale $\lambda = 2\pi/k$ goes non-linear can be given by:

$$t \approx \left[\ln\frac{\phi(0)}{\delta\phi_k(0)}\right]\frac{\lambda}{2\pi}\left(\frac{2}{|K|}\right)^{1/2} \tag{30}$$

For a typical B-ball size of $r \approx (|K|^{1/2}m_s)^{-1}$ this give a hubble constant $H \sim 1/t \approx 0.1|K|m_s$.[5] To find the charge of resulting B-ball, one needs to know the baryon density at above value of hubble constant. If one assumes that the the baryon to photon ratio at current epoch is, $\eta_B \approx 10^{-10}$ then the baryon number assymetry at given value of hubble constant is given by[5]

$$n_B \approx \frac{\eta_B}{2\pi} \frac{H^2 M_p^2}{T_R} \tag{31}$$

where T_R is reheating temperature and $M_p^2 = (8\pi G_N)^{-1}$. This gives the charge enclosed in a region of radius r to be [5]

$$B \approx 10^{15} |k|^{1/2} \frac{\eta_B}{10^{-10}} \frac{10^9 GeV}{T_R} \frac{100 GeV}{m_s} \tag{32}$$

From this one can see that for $|K| \geq .01$, the B-balls of baryon number larger that 10^{14} are likely to be formed. A more accurate analysis applied to the potential appearing in eqn. (28) would give a typical size of B-ball to be of the order 10^{24}[4]. The minimum values of Q would eventually be model dependent.

IV Q-balls: Observational Signatures

Having established the apprximate size of the Q-balls in the last section, we can now look at prospects of observations. To accomplish this, one first needs to know the approximate values for various physical quantities associated with these objects. In this section, first a lower bound on the size of Q-balls would be obtained which would be followed by the estmation of physical quantities and in the end some observational bounds would be discussed.[6]

A small comment on evaporation of Q-balls: As the interior of a Q-ball has a completley filled Dirac sea of fermions; the fermi pressure inhibits decay of Q-matter. Hence, the decay can proceed via surface only and the rate is suppressed by surface-to-volume ratio. Thus large Q-balls would survive longer[4]. In MSSM, the processes that cause Q-ball evaporation are mediated by gauginos and are strongly suppressed if gaugino mass is larger than the mass of the particles forming Q-balls. Thus lifetime of various types of Q-balls are model dependent.

From section II one can get for $U(\phi) \approx constant$

$$\mathcal{E}_\omega \approx a\omega + b/\omega^3 + \omega Q_B \tag{33}$$

where a and b are constants independent of ω. The extremum of the equation can be obtained at $\omega \propto Q_B^{1/4}$ which corresponds to $m_{Q_B} \propto Q_B^{3/4}$. Thus energy of the Q ball can be estimated to be $E_{Q_B} \sim m_\phi Q_B^{3/4}$ while if this object is to dis-integrate into individual baryons with lightest mass m_n, the energy emitted would be $E \sim m_n Q_B$. This gives an lower bound on Q to be

$$Q_B \geq (m_\phi/m_n)^4 \sim 10^8 \tag{34}$$

where we have assumed $m_\phi \sim 100GeV$ the squartk mass and $m_n \sim 1GeV$. For a flat potential of the form $U(\phi) \sim m^4 = constant$ one can further obtain $M_{Q_B} \simeq (4\pi\sqrt{2}/3)mQ_B^{3/4}$, radius $R_Q \sim (1/\sqrt{2})m^{-1}Q_B^{1/4}$ and maximum scalar VEV inside to be $\phi_Q \sim (1/|sqrt2)mQ_B^{1/4}$. These typical values would be used for estmating observational quantities.

If baryon balls make a sizeable contribution to the missing matter of the universe then the their flux seen from earth can be estimated as follows. As seen in the last section, there is only a small window of modes than can grow non-linear for a flat potential and hence one expects the Q balls formed to be of about the same mass. For Q balls making up the galactic halo of density $\rho_D M \approx 0.3GeV/cm^3$ one gets the number density to be

$$n_{_Q} \sim \frac{\rho_{_{DM}}}{M_{_Q}} \sim 5 \times 10^{-5}\, Q_{_B}^{-3/4} \left(\frac{1\mathrm{TeV}}{m}\right) \mathrm{cm}^{-3} \tag{35}$$

If the average velocity of Q balls is assumed to be $v \sim 10^{-3}\, c$. Then the flux is

$$F \simeq (1/4\pi) n_{_Q} v \sim 10^2\, Q_{_B}^{-3/4} \left(\frac{1\mathrm{TeV}}{m}\right) \mathrm{cm}^{-2}\mathrm{s}^{-1}\mathrm{sr}^{-1} \tag{36}$$

For a Kamiokande surafece area of $7.5 \times 10^7 cm^2$ one gets the rate of Q balls going through the detector to be[6]

$$N \sim \left(\frac{10^{24}}{Q_{_B}}\right)^{3/4} \left(\frac{1\mathrm{TeV}}{m}\right) \mathrm{yr}^{-1}. \tag{37}$$

When a nucleon enters outer region of the Q ball, it disscociated into quarks and the energy is released in the form of emitted pions and this forms the basis for the detection of Q balls. One can have two types of Q balls: supersymetric electrically neutral solitons (SENS) and supersymetric electrically charged solitons (SECS) and these two interact very differently. The interaction cross-section would need to be calulcated differently and it would translate differently into the flux rate. Flux for SECS estmated from MACRO search is $F < 1.1 \times 10^{-14}$ cm^{-2} s^{-1} sr^{-1} which translates into a lower bound on baryon number to be $Q_B \geq 10^{21}$ [6]. A similar experiment gives a lower bound for SENS to be $Q_B \geq 3 \times 10^{22}$ for mass $m = 1TeV$. An electically charged Q ball with a smaller baryon number can disintegrate very fast due to efficient dissipation of the energy released. One can also obtain constraints on Q balls from astrophysical observations. A SENS Q ball passing though the earth would lose it velocity due to interaction with matter from earth. But as this change is only tiny fraction of its original velocity, one does not expect Q balls to accumulate inside planets and stars. Neutron stars on the other hand can absorb SENS Q balls due large density of nuclear matter and release a significant amount of energy. SECS can be absorbed by stars as they are stronglly interacting, but, the rate of interaction is not large enough (due to coulomb barrier) to produce a burst of pions which can be detected easily.

Thus Q balls can be considered as candidated for dark matter and they can be detected through strong release of energy along their path if they pass through a detector. The present observational bounds of $Q_B \geq 10^{21}$ are consisent with theoretical estimates of $Q_B \sim 10^{24}$[4]. There would also be possibility of small Q balls forming some portion of dark matter. Any detection of Q balls can put light on early universe cosmolgy as these objects also affect other cosmological processes like baryogenesis and nucleosynthesis.

Appendix

Definition:

Given $\phi(\mathbf{x}) \geq 0, \phi_R(|\mathbf{x}|)$ is the spherically decreasing rearrangement of ϕ if $\phi_R(|\mathbf{x}|)$ is a decreasing function of $|\mathbf{x}|$ and for every non-negative constant M, the Lebesgue measure $\mu[\phi_R(|\mathbf{x}| \geq M] = \mu[\phi(\mathbf{x} \geq M]$ i.e.

$$\int_{\phi > M} d^3\mathbf{x} = \int_{\phi_R > M} d^3\mathbf{x} = V(M) \tag{38}$$

Theorem:

$$\int |\nabla\phi|^2 d^3\mathbf{x} \geq \int |\nabla\phi_R|^2 d^3\mathbf{x} \tag{39}$$

Proof:

Let,

$$I(M) = \int_{\phi>M} |\nabla\phi|^2 d^3\mathbf{x} \tag{40}$$

$$I_R(M) = \int_{\phi_R>M} |\nabla\phi_R|^2 d^3\mathbf{x} \tag{41}$$

Let σ_M define the surface $\phi = M$ and σ_{MR} define the surface $\phi_R = M$. $\nabla\phi$ is normal to the surface σ_M and $\nabla\phi_R$ is normal to the surface σ_{MR}. Differentiating volume $V(M)$, one gets

$$\left|\frac{dV}{dM}\right| = \int \frac{d\sigma_M}{|\nabla\phi|} = \int \frac{d\sigma_{MR}}{|\nabla\phi_R|} \tag{42}$$

Differentiating equations for $I(M)$ and $I_R(M)$ one gets

$$\left|\frac{dI}{dM}\right| = \int |\nabla\phi| d\sigma_M \tag{43}$$

$$\left|\frac{dI_R}{dM}\right| = \int |\nabla\phi_R| d\sigma_{MR} \tag{44}$$

Multiplying and using Schwarz's inequality one gets

$$\left|\frac{dV}{dM}\right|\left|\frac{dI}{dM}\right| = \int \frac{d\sigma_M}{|\nabla\phi|}\int |\nabla\phi| d\sigma_M \geq \left|\int d\sigma_M\right|^2 \tag{45}$$

and

$$\left|\frac{dV}{dM}\right|\left|\frac{dI_R}{dM}\right| = \int \frac{d\sigma_M}{|\nabla\phi_R|}\int |\nabla\phi_R| d\sigma_{MR} = \left|\int d\sigma_{MR}\right|^2 \tag{46}$$

equality as $|\nabla\phi_R|$ is constant on σ_{MR}. We know that sphere has the least surface amongst all surfaces inclosing the same amount of volume. Therefore :

$$\int d\sigma_M \geq \int d\sigma_{MR} \tag{47}$$

$$\Rightarrow \left|\frac{dV}{dM}\right|\left|\frac{dI}{dM}\right| \geq \left|\frac{dV}{dM}\right|\left|\frac{dI_R}{dM}\right| \tag{48}$$

$$\Rightarrow \left|\frac{dI}{dM}\right| \geq \left|\frac{dI_R}{dM}\right| \tag{49}$$

Integrating above expression gives the desired result.

Comment:

By construction, both ϕ and ϕ_R have same potential energy (eqn. (38)) and by above theorem they have different kinetic energies, $T = \int |\nabla\phi|^2$ and hence ϕ_R has less energy which makes it a more stable state.

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