

Unconventional Quantum Hall Effect in Graphene

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Abstract

Graphene is a new two-dimensional material prepared successfully in experiments several years ago. It was predicted theoretically and then observed experimentally to present unconventional quantum Hall effect due to its unique electronic properties which exists quasi-particle excitations that can be described as massless Dirac Fermions. This feature leads to the presence of a Landau level with zero energy in single layer graphene and hence a shift of $\frac{1}{2}$ in the filling factor of the Hall conductivity.

I. Introduction

The integer Quantum Hall Effect (QHE) was discovered by K. von Klitzing, G. Dorda, and M. Pepper in 1980 [1]. It is one of the most significant phenomena in condensed matter physics because it depends exclusively on fundamental constants and is not affected by irregularities in the semiconductor like impurities or interface effects [2]. The presence of the quantized Hall resistance is the reflection of the properties of two-dimensional electron gases (2DEG) in strong magnetic fields, and is a perfect exhibition of basic physical principles. K. von Klitzing was therefore awarded the 1985 Nobel Prize in physics. Further researches on 2DEG led to the discovery of the fractional quantum Hall effect [3]. These achievements make 2DEG become a very vivid field in condensed matter physics.

Since a few single atomic layers of graphite, graphene, was successfully fabricated in experiments recently [4,5], a new two-dimensional electron system has come into physicists' sight. The measurements of the QHE in such a carbon-based electronic material give the unconventional QHE that is different from both integer QHE and fractional QHE discovered before [6,7]. This peculiar phenomenon is due to the unique electronic properties of graphene whose low energy spectrum is governed by Dirac's equation and quasi-particles mimic relativistic particles with zero rest mass. Such a feature leads to the presence of a Landau level with zero energy and hence the anomalous integer QHE. Due to the Dirac nature of its charge carriers and unusual quantum transport properties, graphene is a wonderful material to both basic research such as quantum electrodynamics and new applications in carbon-based electronics.

II. Conventional QHE

A. Classical Hall effect

When a current flows through a rectangular conductor with the area of $L_x L_y$ along its long axis (x axis), the current density is

$$j_x = nqv \quad (1)$$

Where n is the carrier density, q and v are the charge and velocity of carriers. If a magnetic field is applied along the direction (z axis) perpendicular to the plane of the conductor (x-y plane), then the path of carriers will be curved due to a magnetic force, which leads to an accumulation of carriers on one side of the conductor. Consequently, equal and opposite charges appear on the other side and an transverse electric field E_y perpendicular to both directions of the current and magnetic field is established. When $E_y = vB/c$, the forces on carriers induced by the electric field and magnetic field are equal to each other, and the system reaches equilibrium. The defined Hall voltage, Hall resistance and Hall resistivity is

$$U_H \equiv E_y L_y, \quad R_H \equiv \frac{U_H}{I}, \quad \rho_H \equiv \frac{E_y}{j_x} = \frac{B}{nqc} \quad (2)$$

where I is the current flowing through the conductor, and c is the speed of light. By measuring the Hall voltage, we can know the carrier density and the sign of its charge. In addition, the Hall resistivity is proportional to the magnetic field B and inversely proportional to the carrier density n . This effect is known as the classical Hall effect, which is discovered by Edwin H. Hall in 1879. After about one hundred years, K. von Klitzing, G. Dorda, and M. Pepper discovered the quantized Hall effect [1], that is, a series of plateaus appear in the curve of the Hall resistivity versus the magnetic field B or the carrier density n . To understand the quantum Hall effect, it is necessary to have knowledge about the quantum mechanics solution to the problem of the motion of electrons in a homogeneous magnetic field, that is, Landau Levels.

B. Landau levels

If we choose the Landau gauge of the vector potential \mathbf{A} ,

$$A_x = -By, \quad A_y = A_z = 0 \quad (3)$$

then Schrodinger's equation for electrons in a homogeneous magnetic field is

$$H\psi = \frac{1}{2m} \left[\left(p_x + \frac{eB}{c} y \right)^2 + p_y^2 + p_z^2 \right] \psi - \mu s_z B \psi = E \psi \quad (4)$$

where s_z is the z component of the electron spin, and μ is the magnetic moment of electrons [8]. The Hamiltonian, Eq. (4), does not include x and z explicitly, so p_x and p_z are the constant of motion, and we can take the wave function as

$$\psi(x, y, z) = \exp \left[\frac{i}{\hbar} (p_x x + p_z z) \right] \varphi(y) \quad (5)$$

putting it into Eq. (4), we obtain

$$\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + \frac{1}{2} m \omega_c^2 (y - y_0)^2 \right] \varphi(y) = \left(E + \mu s_z B - \frac{p_z^2}{2m} \right) \varphi(y) \quad (6)$$

$$\omega_c = \frac{eB}{mc}, \quad y_0 = -\frac{cp_x}{eB} \quad (7)$$

which is just a shifted harmonic oscillator equation. So we can directly obtain the eigenvalues of the energy

$$E_n = \left(n + \frac{1}{2} \right) \hbar \omega_c - \mu s_z B + \frac{p_z^2}{2m} \quad (8)$$

If electrons are confined in two-dimensions, then the last term in Eq. (8) will vanish. Furthermore, when the system is in a strong magnetic field, electrons are totally spin-polarized so that the second term is a constant. Thus, Eq. (8) gives a series of discrete energy levels known as Landau levels. The degeneracy factor for each Landau level is given by the number of center coordinates y_0 within the sample. For a given sample with the area of $L_x L_y$, the space of center coordinates is

$$\Delta y_0 = \frac{c}{eB} \Delta p_x = \frac{c\hbar}{eB} \Delta k_x = \frac{c\hbar}{eB} \frac{2\pi}{L_x} = \frac{c\hbar}{eBL_x} \quad (9)$$

So the degeneracy factor is

$$N_B = \frac{L_y}{\Delta y_0} = \frac{BL_x L_y}{(ch/e)} = \frac{\Phi}{\Phi_0} \quad (10)$$

where Φ and Φ_0 are magnetic flux and flux quanta, respectively. Additionally, the degeneracy factor per unit area is

$$n_B = \frac{N_B}{L_x L_y} = \frac{eB}{ch} \quad (11)$$

It should be noted that this degeneracy factor for each Landau level is independent of semiconductor parameters like effective mass.

C. Integer quantum Hall effect

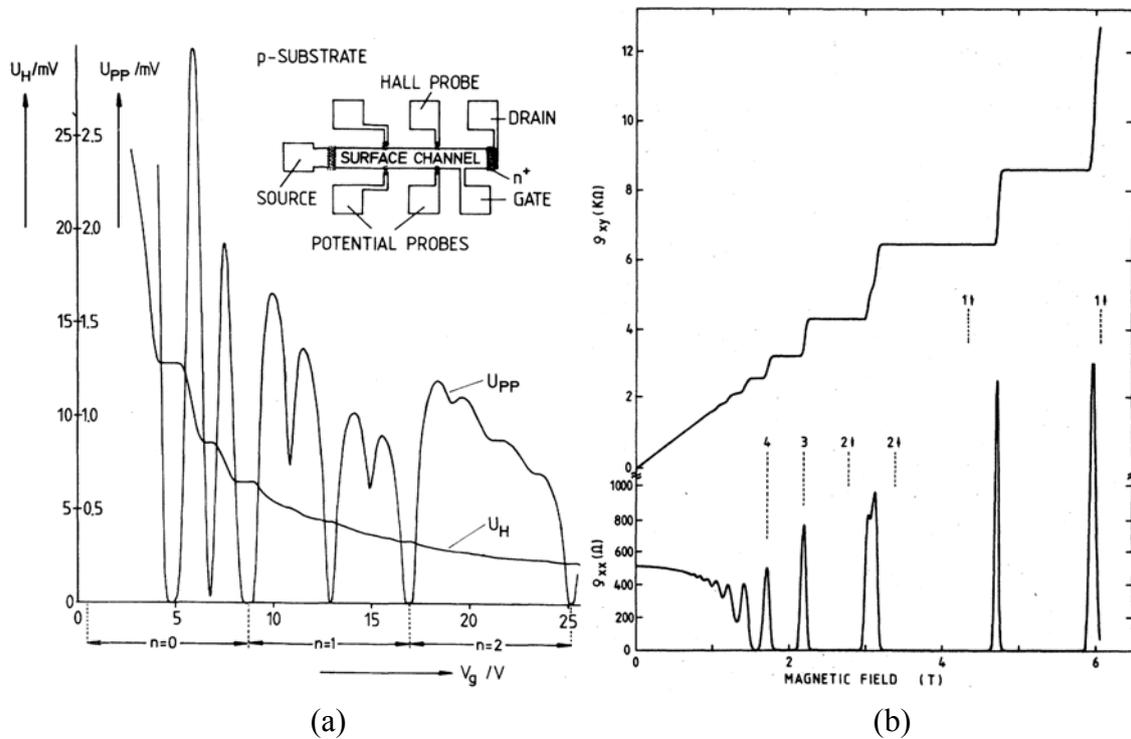


Figure 1: The experimental measurements of the quantum Hall effect. (a) The Hall voltage U_H and the voltage drop between the potential probes, U_{pp} , as a function of the gate voltage V_g in the conditions of $T=1.5$ K and $B=18$ T. The inset shows a schematic diagram of a typical MOSFET device used in the measurement [1]. **(b)** The Hall resistivity $\rho_H = \rho_{xy}$ and longitudinal resistivity ρ_{xx} as a function of the magnetic field at $T=8$ mK and a fixed carrier density corresponding to a gate voltage $V_g=0$ V [2].

In 1980, Klitzing *et al.* found that when the electrons of a sample are well confined in two dimensions (e.g. semiconductor heterostructures) and in the condition of a strong magnetic field and low temperature, the measured Hall resistivity ρ_H (i.e., ρ_{xy}) or Hall voltage U_H is no longer inversely proportional to the carrier density n

(equivalent to a gate voltage V_g applied to the sample) or proportional to the magnetic field B . Instead, a series of plateaus appear in the measured curves (as shown in Fig. 1), that is, the Hall resistance $R_H = \rho_{xy}$ ^① is quantized and it is given by [1,2]

$$R_H = \frac{h}{ie^2} = \frac{25812.8}{i} \Omega, \quad i=1, 2, 3, \dots \quad (12)$$

This effect is called integer quantum Hall effect. Interestingly, the values of the Hall resistance are independent on the materials chosen in the measurements.

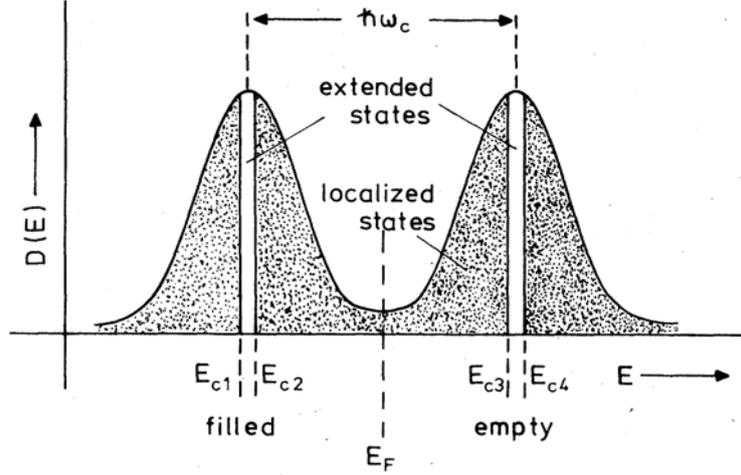


Figure 2: Model for the broadened density of states of a 2DEG in a strong magnetic field. Mobility edges close to the center of the Landau levels separate extended states from localized states [2].

In fact, the quantum Hall effect is characteristic of two-dimensional electron gases in strong magnetic fields. As mentioned in Part II.B, two-dimensional confinement and strong magnetic fields will lead to the emergence of discrete Landau levels. Additionally, in the presence of disorder in real materials Landau levels get broadened and localized states appear between different Landau levels [9] (as shown in Fig. 2). It is noticed that the quantized Hall resistance, Eq. (12), corresponds to the case that i Landau levels are just fully occupied (i.e., $n=in_B$), because from Eq. (2) and (11) we have

$$R_H = \rho_H = \frac{B}{nec} = \frac{B}{in_B ec} = \frac{h}{ie^2} \quad (13)$$

As the magnetic field decreases, the space between two Landau levels ($\hbar\omega_c$) and the degeneracy factor (n_B) will decrease [see Eq. (7) and (11)]. So original i Landau levels cannot offer sufficient states to electrons any more, then extra electrons are supposed to

^① In the two-dimensional case, we have $R_H = \frac{U_H}{I} = \frac{E_y L_y}{j_x L_y} = \frac{E_y}{j_x} = \rho_H = \rho_{xy}$

occupy the $(i+1)$ -th Landau level and this should lead to the continuous change of the Hall resistance. Due to the presence of localized states, however, extra electrons will first occupy these states, and they cannot participate in the conduction of currents. Thus, the Hall resistance does not change (i.e., the existence of plateaus) until the magnetic field is small enough so that the Fermi energy E_F reaches the next Landau level and extra electrons begin to occupy this level. Then, the Hall resistance begins to decrease continuously and stops when this Landau level is fully occupied, as shown in Fig. 1(b).

If we define a filling factor of Landau levels, ν , as the ratio of the surface carrier density n to the degeneracy factor n_B ,

$$\nu = \frac{n}{n_B} = \frac{n}{eB/hc} \quad (14)$$

then the experimental results and above discussion show that when the filling factor is an integer, a quantized Hall resistivity ρ_{xy} is always expected. Besides changing magnetic field, therefore, we can also change carrier density (i.e., changing the gate voltage in experiments) to make the filling factor be an integer and realize QHE [as shown in Fig. 1(a)]. Moreover, in the quantum Hall regime the longitudinal resistivity ρ_{xx} [the resistivity between the source and drain of the device shown in the inset of Fig. 1(a)] becomes immeasurably small [as shown in Fig. 1(b)]. Thus, if we write resistivity as a matrix, then it will have the form as

$$\rho = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} = \begin{pmatrix} 0 & h/ie^2 \\ -h/ie^2 & 0 \end{pmatrix} \quad (15)$$

Since conductivity is the inverse of resistivity, we can obtain

$$\sigma_{xy} = -\frac{\rho_{xy}}{\rho_{xx}^2 + \rho_{yy}^2}, \quad \sigma_{xx} = \frac{\rho_{xx}}{\rho_{xx}^2 + \rho_{yy}^2} \quad (16)$$

By considering the spin degeneracy and possible sublattice degeneracy, we add a factor g to the conductivities and finally obtain their values in the quantized Hall regime

$$\sigma_{xy} = -i \frac{ge^2}{h}, \quad \sigma_{xx} = 0, \quad i = 0, \pm 1, \pm 2, \dots \quad (17)$$

A schematic illustration of the Hall conductivity as a function of the carrier density n in a unit of gn_B is shown in Fig. 3(a).

III. Unconventional QHE in graphene

A. Experimental observations

In 2005, the QHE of graphene was observed in experiments. It is found that differing from other two-dimensional electron system, single layer graphene exhibits an anomalous quantum Hall effect with half-integer filling factors [6,7], and the Hall conductivity is given by

$$\sigma_{xy} = \pm(i + \frac{1}{2}) \frac{ge^2}{h}, \quad i = 0, 1, 2, \dots \quad (18)$$

The measured Hall conductivity and longitudinal resistivity as a function of the carrier density are shown in Fig. 3(b). Compared with Eq. (17) and Fig. 3(a), it is obvious that the Hall conductivity has a shift of $\frac{1}{2}$ in the unit of ge^2/h . To explain and understand this unconventional QHE, it is necessary to recognize the unique electronic structure of graphene which exists quasi-particle excitation behaving as massless Dirac Fermions and their relativistic Landau levels.

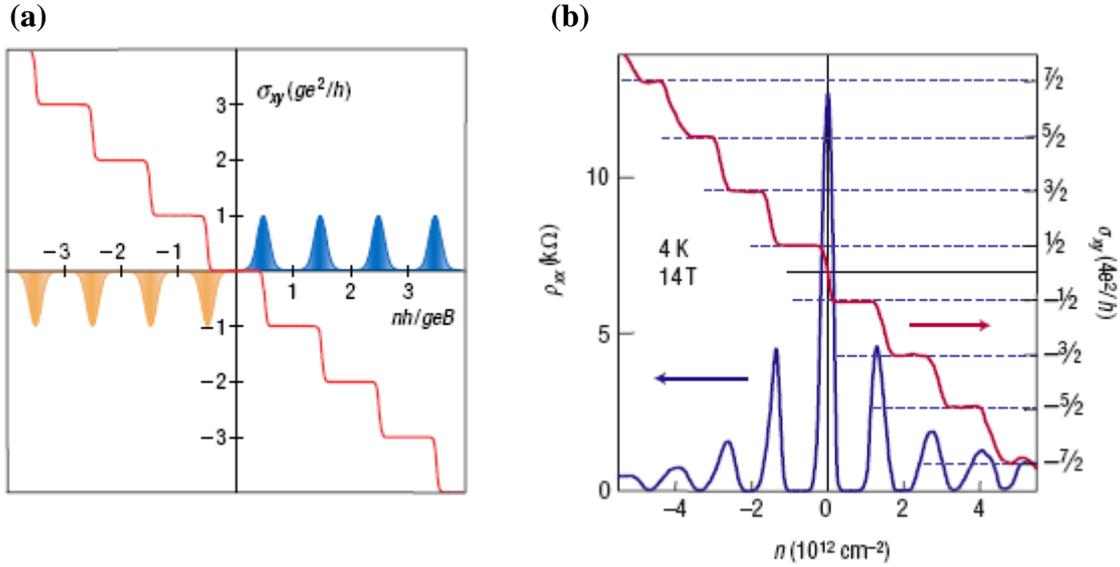


Figure 3: Comparison between conventional and unconventional integer QHE. (a) Schematic illustration of the conventional integer QHE in 2D semiconductor systems by showing the Hall conductivity versus the carrier density, where orange and blue peaks represent the density of state of Landau levels [10]. (b) The experimental measurements of the Hall conductivity and longitudinal resistivity versus the carrier density in graphene [11].

B. Massless Dirac Fermions

Graphene is a single layer of carbon atoms arranged in hexagonal structure and its lattice vectors, reciprocal lattice vectors, and the first Brillouin zone (BZ) are shown in Fig. 4. It is clear that graphene consists of two nonequivalent sublattice A and B, hence there are two nonequivalent corners K and K' in the first Brillouin zone.

A simple tight-binding Hamiltonian for electrons in graphene will give its energy spectrum having the form [12,13]

$$E_{\pm}(\vec{k}) = \pm t\sqrt{3 + f(\vec{k})} - t'f(\vec{k}) \quad (19)$$

$$f(\vec{k}) = 2 \cos(\sqrt{3}k_y a) + 4 \cos\left(\frac{\sqrt{3}}{2}k_y a\right) \cos\left(\frac{3}{2}k_x a\right) \quad (20)$$

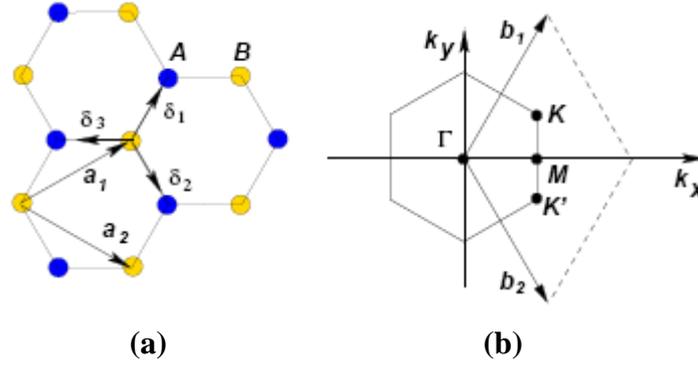


Figure 4: (a) Lattice structure of graphene, made out of two interpenetrating triangular lattices (\mathbf{a}_1 and \mathbf{a}_2 are the lattice unit vectors and δ_i , $i = 1, 2, 3$ are the nearest neighbor vectors). (b) Corresponding Brillouin zone. The Dirac cones are located at the K and K' points. [13]

where a is the carbon-carbon distance, and t and t' are the nearest and next nearest neighbor hopping energy, respectively. Plus and minus signs apply to the upper and lower band, and they are shown in Fig. 5(a). By expanding the full band structure, Eq. (19), close to the K or K' point (named Dirac point) in the first BZ, we obtain a linear dispersion relation

$$E(\vec{q}) \approx \pm v_F \hbar |\vec{q}|, \quad \vec{q} = \vec{k} - \vec{K} \quad (21)$$

where v_F represents the Fermi velocity, given by $v_F = 3ta/2 \sim 10^6$ m/s. It is interesting to notice that the Fermi velocity does not depend on the energy or momentum and the linear dispersion relation will make low energy electron or hole excitations have zero effective mass. DiVincenzo and Mele [14] and Semenoff [15] noticed that even though v_F is much smaller than c , such massless excitations are still governed by a two-dimensional Dirac equation of relativistic quantum mechanics, and its form is given by [16]

$$-i\hbar v_F \vec{\sigma} \cdot \nabla \psi(\vec{r}) = E\psi(\vec{r}) \quad (22)$$

$$\vec{\sigma} = (\sigma_x, \sigma_y), \quad \psi(\vec{r}) = (\psi_A(\vec{r}), \psi_B(\vec{r})) \quad (23)$$

where two components ψ_A and ψ_B are the wave functions related to two sublattices A and B, respectively. Thus, ψ is like a two-component spinor whose spin degree of freedom indicates the sublattices and is called the “pseudospin” to distinguish it from the real electron spin [17]. The massless Dirac function for the momentum around K' has the same form except for replacing (σ_x, σ_y) by $(\sigma_x, -\sigma_y)$ [13].

An immediate consequence of this massless Dirac-like dispersion is a cyclotron mass that depends on the electronic density as its square root [13]

$$m_c = \frac{\sqrt{\pi}}{v_F} \sqrt{n} \quad (24)$$

The experimental data of cyclotron mass is shown in Fig. 5(c). Its well-fitted square root dependence provides evidence for the existence of massless Dirac Fermions in graphene, because the usual parabolic (Schrodinger) dispersion implies a constant cyclotron mass.

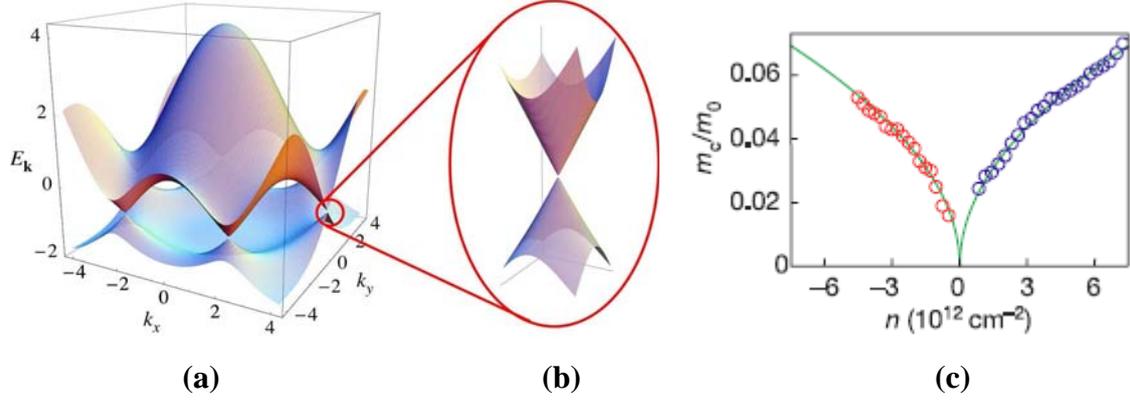


Figure 5: (a) Energy spectrum of graphene in unit of t . (b) Zoom-in of the conical energy bands close to one of the Dirac points [13]. (c) Cyclotron mass m_c of charge carriers as a function of their concentration n , where m_0 is the free electron mass. Positive and negative n correspond to electrons and holes, respectively. Symbols are the experimental data, and solid curves are the best fit to theory [6].

C. Relativistic Landau levels and QHE

Now we consider the problem of a uniform magnetic field B applied perpendicular to the graphene plane. Here we use a simple method in first quantized language [13] to give the solutions to the problem. In the presence of a magnetic field, $-i\hbar\nabla$ in Eq. (22) should be replaced by $-i\hbar\nabla + e\mathbf{A}/c$. If we take the Landau gauge: $\mathbf{A}=B(-y, 0)$ and write the wave function as the form of $\psi(x, y)=\exp(ikx)\phi(y)$, then the Dirac equation will read

$$\hbar v_F \begin{pmatrix} 0 & \partial_y - k + Bey/\hbar c \\ -\partial_y - k + Bey/\hbar c & 0 \end{pmatrix} \phi(y) = E\phi(y) \quad (25)$$

After defining

$$\ell_B = \sqrt{\frac{c\hbar}{eB}}, \quad \omega_c = \sqrt{2} \frac{v_F}{\ell_B}, \quad \xi = \frac{y}{\ell_B} - \ell_B k \quad (26)$$

$$\hat{a} = \frac{1}{\sqrt{2}}(\partial_\xi + \xi), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(-\partial_\xi + \xi) \quad (27)$$

the Eq. (25) becomes

$$\hbar\omega_c \begin{pmatrix} 0 & \hat{a} \\ \hat{a}^\dagger & 0 \end{pmatrix} \phi(\xi) = E\phi(\xi) \quad (28)$$

Since the canonical commutation relation: $[\hat{a}, \hat{a}^+] = 1$ is satisfied, the eigenvectors can be constructed from the eigenfunctions of a harmonic oscillator, φ_n . The obtained eigenvectors and energy eigenvalues are given by [16]

$$\phi_{n,\pm}(\xi) = \begin{pmatrix} \varphi_{n-1}(\xi) \\ \pm \varphi_n(\xi) \end{pmatrix}, \quad E_{n,\pm} = \pm \hbar \omega_c \sqrt{n} = \pm v_F \sqrt{2\hbar e B n / c}, \quad n = 0, 1, 2, \dots \quad (29)$$

These discrete energy eigenvalues are the relativistic Landau levels, where plus and minus sign correspond to electron and hole excitations, respectively. At the opposite Dirac point, K' , we can obtain the same spectrum and hence each Landau level is doubly degenerate. An important difference from the non-relativistic Landau levels is the existence of a zero energy state ($n=0$). It belongs to both electron and hole excitations, and its degeneracy is twice smaller than any other Landau levels with $|n| > 0$ whatever for electrons or holes [18,19]. For a single layer graphene, therefore, when the i -th Landau level is just fully occupied, the relationship between the carrier density and the degeneracy factor of Landau levels is $n=(i+1/2)n_B$, which will lead to the half-integer filling factor in the Hall conductivity of Eq. (18).

IV. Conclusion

In summary, the quantum Hall effect is an exhibition of two-dimensional electron systems at low temperatures and in strong magnetic fields. The formation of discrete Landau levels is crucial to the quantization of the Hall resistance and conductivity. For conventional two-dimensional systems, the plateaus of the Hall resistance appear when the filling factor is an integer. For single layer graphene, however, the Hall conductivity is quantized if the filling factor is a half-integer. This unconventional phenomenon is due to the quasi-particle excitation in graphene which can be described by massless Dirac's equation of the relativistic quantum mechanics. This will lead to the appearance of a Landau level with zero energy and hence a shift of $1/2$ in the filling factor to fulfill the quantization condition of the Hall conductivity.

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