Spontaneous Symmetry Breaking in Non Abelian Gauge Theories

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Abstract

This term paper explores the phenomenon of spontaneous symmetry breaking in systems with non-abelian gauge symmetry. The most famous example comes from the standard model of particle physics, wherein symmetry breaking is responsible for the masses of the W and Z bosons and the corresponding short range of the weak nuclear force. The electroweak theory will be discussed as well as a more ambitious (but less successful) attempt to unify the strong, weak and electromagnetic forces in which symmetry breaking also plays a key role.

1   Introduction

The history of the principle of gauge invariance [9] is a long and storied one that stretches far back into the nineteenth century, beginning with the observation that the electromagnetic scalar and vector potential were underdetermined and thus could be subjected to somewhat arbitrary additional constraint equations (gauge conditions). In 1926, Fock discovered that a charged particle's wave equation was invariant under a position and time-dependent phase transformation $\phi \to \exp(i\theta(\vec{x}, t))\phi$ provided that the electromagnetic potentials were properly adjusted by a gauge transformation to compensate. The idea that the electromagnetic coupling can be looked at as a *consequence* of local phase invariance, which traces back to a speculation made by Weyl in 1928, stands at the center of particle physics today.

In the mid fifties, Yang and Mills [16] showed that the gauge principle could be generalized from phase ($U(1)$) to isospin ($SU(2)$) transformations. The main difficulty associated with this extension is that the isospin transformations do not commute with one another, thus their theory is a termed non-abelian gauge theory, in contrast with the abelian electromagnetism. Their approach is easily generalized from $SU(2)$ to any compact lie group, therefore gauge theories have the allure of associating to an abstract symmetry group of one's choosing a unique theory of interacting matter and gauge fields.

Notwithstanding this allure, these theories at first seemed unsuitable for describing fundamental interactions since they involved massless gauge bosons which had not been observed and were deemed unfit for describing the short-ranged nuclear forces. It turns out this problem can be avoided in two ways: the bosons can become massive due

to spontaneous symmetry breaking or the bosons can not be observed in the particle spectrum due to confinement. Both of these possibilities are realized in the weak and the strong force, respectively, but it is the first that will be the main subject of this paper. The outline is as follows:

First, I will give a brief mathematical introduction to theories with spontaneously broken continuous symmetries, beginning with Goldstone's $U(1)$ model and the abelian Higgs model. This is really nothing more than the Ginzburg-Landau theory learned in class, but it will give me a chance to establish relativistic notation in a familiar context. Then I will generalize to non abelian gauge theories, keeping the mathematical overhead to a minimum.

After the formalism is established I will explore three different theories with different symmetry breaking patterns. The first is $SU(2) \to U(1)$, which is a simple model for electroweak symmetry breaking with no Z boson. This model is not correct, but it is still interesting. For instance, it has magnetic monopole solutions which are an example of topological defects as a consequence of symmetry breaking. The analogue from class would be flux tubes in the $U(1)$ theory.

The second example is $SU(2) \times U(1)_Y \to U(1)_{EM}$. This is the celebrated Glashow-Weinberg-Salam model. I will discuss matter content and the experimental successes of this model. I will also briefly discuss some of its shortcomings as well as alternative symmetry breaking mechanisms to the standard model Higgs.

The third example is $SU(5) \to SU(3) \times SU(2) \times U(1)$. This is Georgi and Glashow's minimal Grand Unified Theory (GUT) attempting to unify the strong, weak and electromagnetic forces.

2 Symmetry Breaking in Gauge Theories

2.1 U(1) theory

For our first example of a theory with a continuous symmetry [5], we consider a complex scalar field ϕ with relativistic action $S = \int d^4x \, \mathcal{L}$ where

$$\mathcal{L} = (\partial^\mu \phi)^* \partial_\mu \phi - V(|\phi|^2).$$

The action is invariant under transformations of the form $\phi \to e^{i\theta}\phi$, so we say the theory has internal $U(1)$ symmetry. The equation of motion (Euler-Lagrange equation) derived from this action is

$$\partial^2 \phi + \phi \frac{\partial V}{\partial |\phi|^2} = 0,$$

which becomes the nice linear Klein Gordon equation for $V = m^2|\phi|^2$. The Hamiltonian is

$$H = \int d^3x \, |\partial_t \phi|^2 + |\vec{\nabla}\phi|^2 + V(|\phi|^2).$$

Notice that the lowest energy field configuration is a constant that minimizes V.

To establish a symmetry breaking minimum, we select a quartic potential $V = \frac{\lambda}{2}(|\phi|^2 - v^2/2)^2$ where v is a real number. The minimum field configuration then

becomes the degenerate circle $|\phi|^2 = \frac{v^2}{2}$. We can then expand the Lagrangian about the arbitrarily selected minimum $\phi_0 = v/\sqrt{2}$ in terms of new fields $\eta(x)$ and $\theta(x)$ where $\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x))e^{i\theta(x)/v}$ and obtain

$$\mathcal{L} = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{(1+\eta/v)^2}{2}\partial_\mu\theta\partial^\mu\theta - \frac{\lambda v^2}{2}\eta^2 + O(\eta^3).$$

Our new fields obey linearized equations of motion

$$(\partial^2 + m_\eta^2)\eta = 0; \qquad \partial^2\theta = 0 \ \text{(exact)}$$

where $m_\eta^2 = \lambda v^2$ and the θ field is massless, the first example of a *Goldstone mode*.

Now, return to the original version of the Lagrangian and consider local $U(1)$ transformations of the form $\phi \to e^{i\theta(x)}\phi$ where the transformation parameter is a function of space-time. Now, the derivative terms in the action are not invariant, but we can compensate by replacing the derivatives with *covariant derivatives* $D_\mu \equiv \partial_\mu - ieA_\mu$ and having the vector field A_μ compensate by transforming $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\theta$. This is an example of the principle mentioned above: a global symmetry can be promoted to a local symmetry with the addition of a vector field and the form of the interactions are determined by this process. Notice that the transformation on A^μ is just a gauge transformation from electricity and magnetism, so we'll identify it with an electromagnetic field and write the Lagrangian:

$$\mathcal{L} = (D^\mu\phi)^* D_\mu\phi - V(|\phi|^2) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor, and the last term was chosen to generate Maxwell's vacuum equations when $\phi = 0$. Notice that the term (and in fact $F_{\mu\nu}$ itself) is gauge invariant, so its addition to the Lagrangian does not break the local symmetry.

We're in position to see the abelian Higgs mechanism first derived in this form in [6]. Consider the same potential as before and expand again about the same ϕ_0, we get

$$\mathcal{L} = \frac{1}{2}\partial^\mu\eta\partial_\mu\eta + \frac{(1+\eta/v)^2}{2}(\partial^\mu\theta - evA^\mu)(\partial_\mu\theta - evA_\mu) - \frac{m_\eta^2}{2}\eta^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + O(\eta^3).$$

We can perform a gauge transformation $A_\mu \to A_\mu + \frac{1}{ev}\partial_\mu\theta$ and obtain

$$\mathcal{L} = \frac{1}{2}\partial^\mu\eta\partial_\mu\eta - \frac{m_\eta^2}{2}\eta^2 + \frac{v^2 e^2}{2}A^\mu A_\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \text{higher order terms.}$$

The linearized equation of motion for the electromagnetic field is therefore $\partial_\mu F^{\mu\nu} = -e^2 v^2 A^\nu$. The Lorenz gauge condition $\partial_\mu A^\mu = 0$, which can be assumed from the outset, is not ruined by the gauge transformation since $\partial^2\theta = 0$. Therefore the equation of motion becomes

$$(\partial^2 + m_\gamma^2)A^\mu = 0$$

where $m_\gamma^2 = e^2 v^2$. The electromagnetic field has obtained a mass.

We've put the Higgs mechanism we learned about in class into the language of relativistic field theory. The main idea is that since a Goldstone boson in a gauge theory is an excitation along an unphysical degree of freedom, it cannot correspond to a physical particle. However, removing the boson from the theory amounts to a choice of gauge. We've now used up our option to choose a gauge that's normally used to make the polarization transverse; the longitudinal polarization is now a physical degree of freedom and the gauge boson can have a mass. This means we are left with a massive η particle (Higgs), a massive gauge boson and no Goldstone boson.

2.2   Yang-Mills Theory

We will now generalize to scalar theories invariant under $SU(N)$ transformations $\phi \to \exp(i\theta_a \tau_a)\phi$ [16] Here, $\phi = (\phi_1, \ldots \phi_N)$ is a complex N-tuple and τ_a are generator matrices, which act as basis vectors for the $SU(N)$ at the identity. In the most common $SU(2)$ case, the index a goes from one to three and $\tau_a = \frac{1}{2}\sigma_a$ where σ_a are the Pauli matrices. The generators can be chosen [2] to obey $\mathrm{tr}(\tau_a) = 0$, $\mathrm{tr}(\tau_a \tau_b) = \frac{1}{2}\delta_{ab}$ and $[\tau_a, \tau_b] = if_{abc}\tau_c$ where the f_{abc} is a real, antisymmetric tensor. In the familiar $SU(2)$ case, $f_{abc} = \epsilon_{abc}$.

We can take a similar Lagrangian for the scalar field as above $\mathcal{L} = |\partial\phi|^2 - V(|\phi|^2)$ (where Lorentz and $SU(N)$ index sums are implied) which has the required invariance. To promote the symmetry to a local symmetry, we must replace the derivatives with covariant derivatives $D_\mu = \partial_\mu - igA_\mu$. The gauge field must transform like $A_\mu \to \exp(i\theta_a \tau_a)(A_\mu + \frac{1}{g}\partial_\mu \theta_b \tau_b)\exp(-i\theta_a \tau_a)$ so that $D_\mu \phi \to \exp(i\theta_a \tau_a)D_\mu \phi$. Notice that the gauge field is now an $N \times N$ matrix. It can be taken to be traceless and Hermitian (notice the gauge transformation doesn't change this property), so it can be written in terms of its components $A_\mu = A_{a\mu}\tau_a$.

All that remains is to come up with dynamics for the gauge field alone, the analogue of the Maxwell Lagrangian $-\frac{1}{4}(F_{\mu\nu})^2$. It turns out [10] that the key property of the field tensor that generalizes is that $-ieF_{\mu\nu} = [D_\mu, D_\nu]$. In the non-abelian case have $[D_\mu, D_\nu] = -ig(\partial_\mu A_\nu - \partial_\nu A_\mu - g^2[A_\mu, A_\nu]$. The second term, $-g^2[A_\mu, A_\nu] = -ig^2 A_\mu^a A_\nu^b f^{abc}\tau^c$ was not present in the $U(1)$ case. The analogue of the field tensor can then be defined $F_{\mu\nu} = F_{\mu\nu}^a \tau^a$ where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c.$$

The Lagrangian for a non abelian gauge field coupled to a scalar N-tuple is then defined as

$$\mathcal{L} = (D_\mu \phi)^\dagger D^\mu \phi - V(\phi^\dagger \phi) - \frac{1}{4}F^{a\mu\nu}F_{\mu\nu}^a$$

The final term is gauge invariant since it can be written $-\frac{1}{2}\mathrm{tr}(F^{\mu\nu}F_{\mu\nu})$ and the field tensor transforms like $F_{\mu\nu} \to \exp(i\theta_a \tau_a)F_{\mu\nu}\exp(-i\theta_a \tau_a)$. This term can be justified not only because it looks similar to the Maxwell Lagrangian, but also since it is the only renormalizable, Lorentz-and-gauge–invariant operator that obeys time-reversal symmetry.

The gauge field part of the Lagrangian can be expanded to

$$-\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 - \frac{1}{2}gf^{abc}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)A^{b\mu}A^{c\nu} - \frac{1}{4}g^2(f^{abc}A_\mu^b A_\nu^c)^2.$$

This Lagrangian has quartic and cubic terms in the A^a_μ so unlike the maxwell Lagrangian, it contains interactions whose forms are fixed by symmetry. There is no mass term of the form $m^2(A^a_\mu)^2$ so the gauge fields are massless and a term of such a form would violate gauge invariance just like in the $U(1)$ case.

Symmetry Breaking Before we discuss symmetry breaking, we will make one slight generalization. There is no reason why the matter fields need to transform under the fundamental representation of $SU(N)$. In fact, everything we did above holds for fields transforming under an arbitrary representation. The generators will change (perhaps to matrices of a different size), but they will still be traceless and Hermitian and obey the commutation relations (these are properties of special unitary transformations in general).

If the matter field's potential is chosen so that spontaneous symmetry breaking occurs, we can see how the gauge bosons will obtain mass [8]. Lets, for instance, take *phi* in the fundamental representation of $SU(2)$. Let's say there is a potential as before of $V(\phi^\dagger\phi) = \frac{\lambda}{2}(\phi^\dagger\phi - \frac{v^2}{2})^2$. This yields the ground state condition $\phi^\dagger\phi = v^2/2$. We can choose $\phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}$. Plugging $\phi = \phi^0$ this into the term $|D_\mu\phi|^2 = |(\partial_\mu - igA^a_\mu\tau^a)\phi|^2$, we get a term quadratic in the gauge fields $g^2A^a_\mu A^{b\mu}\phi_0^\dagger\tau^a\tau^b\phi_0 = \frac{1}{2}(m^2)^{ab}A^a_\mu A^{b\mu}$. The quantity

$$(m^2)^{ab} = 2g^2\phi_0^\dagger\tau^{(a}\tau^{b)}\phi_0$$

is known as the mass matrix for the gauge bosons. Diagonalizing it give the linear combinations of gauge fields that have definite masses. Notice we took the symmetric part of the matrix since it's multiplied by the symmetric $A^a_\mu A^{b\mu}$. In our current example,

$$(m^2)^{ab} = 2g^2\phi_0^\dagger\frac{1}{4}\delta^{ab}\phi_0 = \frac{1}{4}(gv)^2\delta^{ab}.$$

In this example the mass matrix is diagonal so each of the three gauge bosons A^1_μ, A^2_μ and A^3_μ hs a mass $m_A = gv/2$.

We will see below that the symmetry group that leaves ϕ_0 invariant is important. In the present example and the $U(1)$ example above the classical minimum had no residual symmetry and so we say the symmetry breaking patterns are $SU(2) \rightarrow$ nothing and $U(1) \rightarrow$ nothing. We will see that the lack of residual symmetry means that all of the gauge fields get mass and that breaking patterns with residual symmetry have gauge particles that remain massless.

3 $SU(2) \rightarrow U(1)$

The first example we will discuss in detail was introduced by Georgi and Glashow in 1972 [3] as a model of electroweak symmetry breaking. In this case, our Higgs field transforms under the three-dimensional vector representation of $SU(2)$. This representation is familiar as the defining representation of $SO(3)$. The Hermitian, traceless

generator matrices are given by

$$\tau^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \tau^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad \tau^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

In other words, $(\tau^a)^{bc} = -i\epsilon^{abc}$. Noticing that this is a real representation, w can pick a Lagrangian

$$\mathcal{L} = \frac{1}{2}(D_\mu \phi)^2 - \frac{\lambda}{8}(\phi^\dagger \phi - v^2)^2 - \frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}.$$

We can pick an average value $\phi_0 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$. Notice that this value of ϕ_0 has residual symmetry: it is invariant under rotations about the z axis. This also follows from the fact that $i\tau^3\phi_0 = 0$.

We can immediately compute the mass matrix for this theory

$$(m^2)^{ab} = g^2\phi_0^\dagger \tau^a \tau^b \phi_0 = v^2(\tau^a)^{3c}(\tau^b)^{c3} = -v^2\epsilon^{a3c}\epsilon^{bc3} = (gv)^2(\delta^{ab} - \delta^{a3}\delta^{b3}).$$

So we see $m_1 = m_2 = gv$ and $m_3 = 0$. One of our bosons is massless and the other two are massive. These can be interpreted as a photon and two massive bosons mediating the weak force.

We can parameterize the fluctuations of the real scalar field about its ground state by

$$\phi = \exp(i\theta^1(x)\tau^1 + i\theta^2(x)\tau^2)\begin{pmatrix} 0 \\ 0 \\ v + \eta(x) \end{pmatrix}.$$

Just like in the $U(1)$ case, a gauge transformation can then be performed to eliminate the exponential pre-factor, retaining only the longitudinal fluctuation. The Lagrangian then becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)^2 - \frac{\lambda v^2}{2}\eta^2 - \frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \frac{g^2v^2}{2}(A^{1\mu}A^1_\mu + A^{2\mu}A^2_\mu) - \frac{\lambda}{8}\eta^4 - \frac{\lambda v}{4}\eta^3 + \frac{2\eta v + \eta^2}{2}(A^{1\mu}A^1_\mu + A^{2\mu}A^2_\mu).$$

This has a massive scalar Higgs field η with mass $\sqrt{\lambda v^2}$, two massive and one massive gauge field, as well as Higgs self-interactions, gauge-gauge interactions from the pure Yang-Mills term and interactions between the Higgs and gauge bosons.

Monopoles The $SU(2) \to U(1)$ model is not the correct theory of electroweak symmetry breaking. The most obvious problem is the omission of the Z boson and the corresponding neutral current weak interactions. However, the theory has one interesting feature that is not present in the correct GWS theory: monopole solutions discovered independently by 't Hooft and Polyakov in 1974 [14]. The idea is that if the massless generator is identified with a photon, one can find stable solutions of the classical field equations with magnetic charge. This is a direct consequence of spontaneously broken gauge symmetry as we shall see.

The problem is similar to magnetic flux tubes in a superconductor. The feature is that the phase of the order parameter winds around as you transit a circle enclosing the defect. In this three-dimensional analogue, we take the Higgs field to obey

$$\vec{\phi} = v\hat{r}$$

at spatial infinity. Normally, the fact that the field is changing at infinity would cause the energy of this solution to diverge thanks to the gradient term, but we can arrange the gauge field to cancel this and cause $D^\mu \vec{\phi} \to 0$ at infinity by putting

$$A_i^a = \frac{1}{e} \epsilon^{bij} \frac{r^j}{r^2}$$

(this is a static solution so the fields are time independent and A_0^a can be taken to vanish.)

Now, we need an electromagnetic field tensor. This is given by the gauge invariant definition

$$G_{\mu\nu} = \frac{F_{\mu\nu}^a \phi^a}{|\phi|} - \frac{\epsilon^{abc} \phi^a (D_\mu \phi)^b (D_\nu \phi)^c}{g|\phi|^3}.$$

In a small region at spatial infinity where the Higgs field can be taken to be constant,

$$G_{\mu\nu} = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3.$$

Plugging in the asymptotic values gives $F_{ij} = \frac{\epsilon^{ijk} r^k}{gr^3}$ which corresponds to a radial magnetic field

$$\vec{B} = \frac{\hat{r}}{gr^2}.$$

Since this result for the charge of the monopole is not analytic in g, this is a non-perturbative property.

The energy of this solution (mass of the monopole is

$$E = \int d^3x \left(\frac{1}{4}(F_{ij})^2 + \frac{1}{2}(D_i\phi)^2 + V \right) > 4\pi v/g = 4\pi m/g^2 \approx 137m$$

where m is the mass of one of the gauge bosons. So the monopole's mass is a couple orders of magnitude above the mass scale associated with the symmetry breaking.

4 $SU(2) \times U(1) \to U(1)$[15]

The $SU(2) \to$ nothing breaking pattern mentioned above yields no massless gauge bosons and so cannot incorporate the photon, but if the gauge symmetry is extended to include an additional $U(1)$ factor, the ϕ_0 we choose will remain invariant under a one parameter group of transformations and we will have residual symmetry and one massless boson. Matter fields are classified by which representation of $SU(2)$ they transform under and what *hypercharge* they carry under the $U(1)$ transformation. For instance if the field transforms $\psi \to \exp(iY\theta)\psi$ under a $U(1)$ rotation of θ, the hypercharge is given by that Y value.

The Higgs field is a complex double that transforms as $(2, 1/2)$, which is code for it transforming under the two-dimensional defining representation of $SU(2)$ and having $Y = 1/2$. We will again set a potential that forces $|\phi_0|^2 = v/2$ and arbitrarily set $\phi_0 = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$. Then the covariant derivative can be worked out explicitly

$$D_\mu \phi_0 = (\partial_\mu - igA_\mu^a \tau^a - ig'YB_\mu)\phi_0 = -i/2(A_\mu^a \sigma^a + B_\mu^a)\begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = -\frac{iv}{2\sqrt{2}}\begin{pmatrix} gA_\mu^1 - igA_\mu^2 \\ g'B_\mu - gA_\mu^3 \end{pmatrix}$$

which yields

$$(D^\mu \phi_0)^\dagger D_\mu \phi_0 = \frac{v^2}{8}(g^2(A_\mu^1)^2 + g^2(A_\mu^2)^2 + (gA_\mu^3 - g'B_\mu)^2).$$

This is our first example of a mass matrix with off-diagonal terms:

$$m^2 = \frac{v^2}{4}\begin{pmatrix} g^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2 & -gg' \\ 0 & 0 & -gg' & (g')^2 \end{pmatrix}$$

The determinant of this matrix is zero, showing that there is a massless boson. The first two components both have mass, as well as the linear combination $gA_\mu^3 - g'B^\mu$. The conventional choices for the orthogonal linear combinations that diagonalize m^2 are

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(A_\mu^1 \pm A_\mu^2), \quad Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(gA_\mu^3 - g'B_\mu), \quad A_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(g'A_\mu^3 + gB_\mu)$$

which have mass eigenvalues $M_W = gv/2$, $M_Z = \sqrt{g^2 + g'^2}v/2$ and $M_A = 0$. The gauge field part of the covariant derivative can be rewritten:

$$gA_\mu^a \tau^a + g'YB_\mu = \frac{g}{\sqrt{2}}(W_\mu^+ \tau^+ + W_\mu^- \tau^-) + \frac{1}{\sqrt{g^2 + g'^2}}Z_\mu(g^2\tau^3 - g'^2Y) + \frac{gg'}{\sqrt{g^2 + g'^2}}A_\mu(\tau^3 + Y)$$

with $\tau^\pm = \tau^1 \pm \tau^2$ This shows that the electric charge of a particle is given by $e(\tau^3 + Y)$ where $e = \frac{gg'}{\sqrt{g^2 + g'^2}}$. In other words, the particle's charge depends on its weak isospin and hypercharge. For instance, the upper component of the Higgs doublet has electric charge $1/2 + 1/2 = 1$ and the bottom component has charge $-1/2 + 1/2 = 0$.

Experiments[1]   Generally, particle masses and coupling constants are input parameters of a theory and can't be predicted. However, the situation is somewhat different in a theory with spontaneously broken symmetry. Since the masses of the bosons are derived from the same input parameters, the theory actually makes concrete predictions about the relationship of the masses in terms of experimentally measured quantities.

The Weinberg mixing angle θ_W is defined by $\tan(\theta_W) = g'/g$. This angle which is a function of the couplings can be measured by experiment. Due to radiative corrections,

its value changes with the energy scale involved and its most precisely measured value is $\sin^2(\theta_W) = .2313$ at the energy 91.2 GeV (the mass of the Z boson). We have the formula $M_W = M_Z \cos\theta_W$ which agrees well with the experimental values of $M_Z = 91.19$ and $M_W = 83.98$. The independent parameters measured to high precision are the Z boson mass, the Fermi constant G_F and the fine structure constant α, from which θ_W can be derived. The predicted value of M_W agrees with experiment very well when radiative corrections are taken into account.

The GWS theory's prediction of weak neutral currents was verified in 1973, and the W and Z bosons were directly discovered ten years later (both in experiments at CERN). This and the aforementioned high-precision agreements of the mass relationships with theory make the GWS theory an experimental success to be sure.

However, the theory predicts another particle, the famous Higgs boson whose mass depends on the unknown parameter λ (the Higgs self coupling), so cannot be predicted from the known quantities above. As of right now the particle has not been discovered, though it's mass has been narrowed down substantially and a promising program is underway at the LHC.

Fermions   The parity-violating nature of the weak interaction means that the interaction treats left-handed and right-handed particle fields differently. In fact, only the left-handed fields

$$\begin{pmatrix} \nu \\ e \end{pmatrix}_L \quad \begin{pmatrix} u \\ d \end{pmatrix}_L$$

transform nontrivially under the $SU(2)$ group. The formula $Q = \tau^3 + Y$ implies that the neutrino-electron doublet has $Y = -1/2$ to make the neutrino neutral and the electron have $Q = -1$. Similarly, the quark doublet has $Y = +1/6$. The right-handed fields are all arranged in separate singlet representations where the hypercharge is necessarily equal to the electric charge of the particle. For instance, the right-handed down-quark transforms as $(1, -1/3)$.

The impact of symmetry breaking on fermions is to give them mass. Explicit mass terms are forbidden in chiral gauge theories since they mix left-handed and right-handed fields. To get around this problem, the masses are introduced in by Yukawa couplings to the Higgs boson. For instance, for the lepton field, we have the term $\lambda_e \bar{e}_L \phi e_R$ which, for $\phi = \phi_0$ gives an electron mass term (but not a neutrino mass). There is a similar, but slightly more complicated Yukawa coupling for the quark fields that gives them their masses. These Yukawa couplings make up a large portion of the free parameters of the standard model.

Alternatives   Despite the success of the GWS theory, it has some shortcomings that have led some to suggest alternatives to the model suggested above. Although the experimental evidence suggests that the symmetry breaking pattern above occurs, there's no guarantee that it is driven by a fundamental scalar field. One popular alternative is Technicolor [12] wherein the symmetry breaking field is dynamically generated, as it is in superconductivity. The condensate driving the symmetry breaking may be from known or hypothetical particles. One appeal of this mechanism over the fundamental scalar is that it avoids hierarchy problem, which is that radiative

corrections to the Higgs mass have very large natural values which must be precisely cancelled in order to have the Higgs mass on the order of the electroweak scale.

5 $SU(5) \to SU(3) \times SU(2) \times U(1)$

The GWS model shows the promise of non abelian gauge theories for unifying the fundamental forces. It would be aesthetically nice if the standard model was the low-energy phenomenology of a larger gauge theory, a Grand Unified Theory. The energy scales could be separated by symmetry breaking. Georgi and Glashow proposed a theory to that effect in 1974 [4] based on the $SU(5)$ gauge group. The Higgs field required transforms under the adjoint representation of $SU(5)$ and as such can be represented by a 5×5 traceless hermitian matrix. If the vacuum value is

$$\phi_0 \propto v\text{diag}[2, 2, 2, -3, -3],$$

this value commutes with an $SU(3)$ subgroup $\begin{pmatrix} T_a & 0 \\ 0 & 0 \end{pmatrix}$ where T_a are the 3×3 $SU(3)$ generators as well as an $SU(2)$ subgroup and a one-parameter $U(1)$ subgroup corresponding to the generator proportional to ϕ_0. Since, in the adjoint representation, the mass matrix is proportional to $g^2\text{tr}[\tau^a, \phi_0][\tau^b, \phi_0]$ [10], the commuting generators correspond to massless gauge fields. In this case there are twelve massless fields, corresponding to the generators of $SU(3) \times SU(2) \times U(1)$ and twelve massive generators whose mass scale $M_{\text{GUT}} \approx 10^{15}$ GeV is set by parameters in a symmetry breaking potential of the form [11]

$$V(\phi) = m^2\text{tr}(\phi^2) + a(\text{tr}(\phi^2))^2 + b\text{tr}(\phi^4).$$

This adjoint Higgs particle only accomplishes the separation of the GUT scale. Another Higgs field, transforming as a 5 (fundamental) provides electroweak symmetry breaking.

A single generation of the standard model fits perfectly into two irreducible representations of $SU(5)$, the $\bar{5}$ and 10

$$
\begin{pmatrix} d^1 \\ d^2 \\ d^3 \\ e \\ \nu \end{pmatrix}_R
\qquad
\begin{pmatrix}
0 & \bar{u}_3 & -\bar{u}_2 & u_1 & -d_1 \\
-\bar{u}_3 & 0 & \bar{u}_1 & -u_2 & -d_2 \\
\bar{u}_2 & -\bar{u}_1 & 0 & -u_3 & -d_3 \\
u_1 & u_2 & u_3 & 0 & -\bar{e} \\
d_1 & d_2 & d_3 & \bar{e} & 0
\end{pmatrix}_L
$$

This perfect fit could be improved aesthetically in only one way: by unifying into a single irreducible representation. In fact, the 16 dimensional representation of $Spin(10)$ works [7], provided that one additional matter particle, the sterile right-handed neutrino, is introduced.

The massive gauge bosons in the $SU(5)$ theory mix the quark and lepton fields, leading to the prediction of proton decay. The proton lifetime has an experimental lower bound of around 10^{34} years [1] from the Super-K experiment, which is above the theoretical prediction of 10^{32} years of $SU(5)$.

10

Additionally, the theory has magnetic monopoles with mass $\approx 137 M_{\mathrm{GUT}}$. Monopoles and other topological defects such as textures, domain walls and cosmic strings are generic predictions of Grand Unified theories, but they have not been found. In fact, one of the major motivations for inflationary cosmology (Alan Guth, 1981) was to explain the paucity of magnetic monopoles in the universe.

In addition to lack of experimental evidence, GUTs show some theoretical shortcomings. For one, they don't improve the Hierarchy problem of the standard model. In fact, they contribute radiative corrections of order M_{GUT} to the Higgs mass, which is some thirteen orders of magnitude larger than the electroweak scale, requiring delicate cancellation. Also $SU(5)$ gives no insight into the reason for the different generations of elementary particles, nor to the sloppy assortment of fermion masses. In fact, $SU(5)$ predicts quark-to-lepton mass ratios that don't agree with experiment.

6   Conclusion

Non-Abelian gauge theories play a central role in theoretical particle physics because they are extremely rich while also being highly constrained by symmetry. They are a natural choice for fundamental theories in which naturalness is a priority. Spontaneously broken gauge symmetry makes the structure richer, loosening the constraints set by manifest gauge invariance and allowing massive vector particles. Though I stuck mainly to the classical theory and didn't mention quantization very much, it should be emphasized that one of the great triumphs of these theories is that they are renormalizable [13] and gauge invariance is an essential property for this. In fact, theories with massive vector bosons put in by hand do not have this desired property.

The standard model of particle physics is a non-abelian gauge theory and its fantastic success established gauge invariance and spontaneous symmetry breaking as major ingredients in realistic theories. Although the model is incomplete and the exact mechanism of symmetry breaking is still an open question until the Higgs hunt ends, the top quark, W and Z represent the most recent major successful predictions in particle physics. Although attempts at further unification have not met success so far, there's little doubt that whatever the answer is, spontaneous symmetry breaking will play a role.

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