Bosonization of 1-Dimensional Luttinger Liquid

Di Zhou

December 13, 2011

Abstract

Due to the special dimensionality of one-dimensional Fermi system, Fermi Liquid Theory breaks down and we must find a new way to solve this problem. Here I will have a brief review of one-dimensional Luttinger Liquid, and introduce the process of Bosonization for 1-Dimensional Fermionic system. At the end of this paper, I will discuss the emergent state of charge density wave (CDW) and spin density wave (SDW) separation phenomena.

1 Introduction

3-Dimensional Fermi liquid theory is mostly related to a picture of quasi-particles when we adiabatically switch on interactions, to obtain particle-hole excitations. These quasi-particles are directly related to the original fermions. Of course they also obey the Fermi-Dirac Statistics. Based on the free Fermi gas picture, the interaction term: (i) it renormalizes the free Hamiltonians of the quasi-particles such as the effective mass, and the thermodynamic properties; (ii) it introduces new collective modes. The existence of quasi-particles results in a finite jump of the momentum distribution function $n(k)$ at the Fermi surface, corresponding to a finite residue of the quasi-particle pole in the electrons' Green function.
1-Dimensional Fermi liquids are very special because they keep a Fermi surface (by definition of the points where the momentum distribution or its derivatives have singularities) enclosing the same k-space volume as that of free fermions, in agreement with Luttingers theorem.

1-Dimensional electrons spontaneously open a gap at the Fermi surface when they are coupled adiabatically to phonons with wave vector $2k_F$. The mean-field theory tells us that there is a charge or spin density wave instability at some finite temperature for repulsive interactions implying that there can be no Fermi liquid in 1-Dimension. There are no fermionic quasi-particles, and their elementary excitations are rather bosonic collective charge and spin fluctuations dispersing with different velocities. An incoming electron decays into such charge and spin excitations which then spatially separate with time (charge-spin separation). The correlations between these excitations are anomalous and show up as interaction-dependent nonuniversal power-laws in many physical quantities where those of ordinary metals are characterized by universal (interaction independent) powers.
In this paper I had a brief review on how the 1-Dimensional Fermi system comes out, and how spin-charge separation emerges in 1-Dimensional Fermi system. By using the particle-hole excitation operators, which is the density operators for Fermions, (the so-called bosons), we can diagonalize this Hamiltonian. Therefore we can obtain the grounp velocity for both system of CDW and SDW, and finally find their difference on velocities, and the two waves seperate.

2 Luttinger Liquid

Before looking at the Luttinger Liquid, we want to have a look at 1-D free Fermion system first. The lattice Hamiltonian for a non-interacting hopping Fermion system is written as:

$$H_f = -t \sum_{n\sigma} (\psi^+_{n\sigma} \psi_{n+1\sigma} + h.c.) + \mu \sum_{n\sigma} \psi^+_{n\sigma} \psi_{n\sigma} \tag{1}$$

where t is the hopping constant between nearest lattice sites, and μ is the chemical potential, and σ is the spin degree of freedom. Using the Fourier Transformation into the momentum space, we can find the eigenvalue for this Hamiltonian, is

$$\epsilon_k = -2t \cos ka + \mu \tag{2}$$

where $\mu = 0$ for $k_F = \pi/2a$, or, we can simply absorb the chemical potential μ when expanding around the chemical potential.
The Luttinger Liquid can be derived from the 1-D Hubbard model

when the above free Fermion system includes an interaction term,

$$H = H_f + H_U = H_f + U \sum_n \rho_{n\uparrow}\rho_{n\downarrow} \tag{3}$$

for temperature much lower than the Fermi energy, we want to expand the energy around the Fermi points. By writing the site amplitudes for right and left moving components, where $k_F = \frac{\pi}{2a}$

$$\psi_{n\sigma} = e^{ik_F na}\psi_{n\sigma+} + e^{-ik_F na}\psi_{n\sigma-} = R_\sigma(n) + L_\sigma(n) \tag{4}$$

the assumption here for left and right moving Fermions is, they are slow varing in space, and have mean momemtum value of $\pm k_F$. Therefore, we can expand the Fermion field for position $(n+1)a$ at the center of the position na:

$$\psi_{n+1\sigma\pm} = \psi_{n\sigma\pm} + a\partial_x\psi_{n\sigma\pm} + ... \tag{5}$$

plug this term back into Eq.(4), retaining only the first order of a. Since

$$\begin{aligned} \exp(ik_F a) + \exp(-ik_F a) &= 0 \\ \exp(ik_F a) - \exp(-ik_F a) &= 2i \\ \sum_n \exp(ik_F a(2n+1)) &= 0 \end{aligned} \tag{6}$$

the third equation vanishes because only $x = 2n\pi$ suvives in the summation of $\sum_m \exp(imx)$. Taking the notation that $a\sum_n \to \int_{-L/2}^{L/2} dx$, $\psi_{n\sigma\pm} = \sqrt{a}\psi_{\sigma\pm}(x)$, now the 1-D free Fermi Hamiltonian is simplified into

$$H_f = -\hbar v_F \sum_\sigma \int_{-L/2}^{L/2} \left[\psi^+_{\sigma+}(i\partial_x)\psi_{\sigma+} + \psi^+_{\sigma-}(-i\partial_x)\psi_{\sigma-}\right] dx \tag{7}$$

where we define $\hbar v_F = 2at$ here. Fourier Transform it into the momentum space,

$$H_f = \hbar v_F \sum_\sigma \int kdk \left[\psi^+_{\sigma+}(k)\psi_{\sigma+}(k) - \psi^+_{\sigma-}(k)\psi_{\sigma-}(k)\right] \tag{8}$$

or, equivalently, Luttinger performed a canonical transformation of the form

$$\begin{aligned} &\psi_{\sigma+}(k) = b_{k\sigma} \quad \psi_{\sigma-}(k) = c^+_{k\sigma} \quad k > 0 \\ &\psi_{\sigma+}(k) = c^+_{k\sigma} \quad \psi_{\sigma-}(k) = b_{k\sigma} \quad k < 0 \end{aligned} \tag{9}$$

in the sense of particle-hole transformation. The transformed normal-ordered free Hamiltonian is written as:

$$H_f = \hbar v_F \int_{-\infty}^{\infty} |p| dp \left[b_{p\sigma}^+ b_{p\sigma} + c_{p\sigma}^+ c_{p\sigma} \right] \tag{10}$$

On the other hand, the interaction part of the Hamiltonian can also be rewrite in momentum space. First let us rewrite the density operator in momentum space:

$$\begin{aligned} \rho_{\sigma\pm}(k) &= \int_{-\infty}^{\infty} dx\, e^{-ikx} \int_{-\infty}^{\infty} \frac{dp}{2\pi}\frac{dq}{2\pi} \psi_{\sigma\pm}^+(p)\psi_{\sigma\pm}(q) e^{i(q-p)x} \\ &= \int_{-\infty}^{\infty} dp\, \psi_{\sigma\pm}^+(p)\psi_{\sigma\pm}(p+k) \end{aligned} \tag{11}$$

From the definition of Eq.(4), we find the relation between $\rho_{n\sigma\pm}(x)$ and $\rho_{n\sigma}(x)$:

$$\begin{aligned} \rho_{n\sigma} &= \psi_{n\sigma}^+ \psi_{n\sigma} \\ &= \left(e^{-ik_F na} \psi_{n\sigma+}^+ + e^{ik_F na} \psi_{n\sigma-}^+ \right) \left(e^{ik_F na} \psi_{n\sigma+} + e^{-ik_F na} \psi_{n\sigma-} \right) \\ &= \psi_{n\sigma+}^+ \psi_{n\sigma+} + \psi_{n\sigma-}^+ \psi_{n\sigma-} \\ &= \rho_{n\sigma+} + \rho_{n\sigma-} \Rightarrow \rho_\sigma(k) = \rho_{\sigma+}(k) + \rho_{\sigma-}(k) \end{aligned} \tag{12}$$

where

$$\rho_{\sigma s}(k) = \int dp\, \psi_{\sigma s}^+(p+k)\psi_{\sigma s}(p) \tag{13}$$

The interaction part, if we don't consider the spin-coupling terms, could be written as:

$$H_{\text{int}} = \frac{1}{2L} \sum_{\sigma s} \int \frac{dq}{2\pi} \left(g_4 \rho_{\sigma s}(q) \rho_{\sigma s}(-q) + g_2 \rho_{\sigma s}(q) \rho_{\sigma \bar{s}}(-q) \right) \tag{14}$$

we can also obtain $\rho_{\sigma\pm}(k)$ operators in terms of b and c operators. From our heuristic argument above, suggesting charge density modulations to be the basic excitations of the system, we expect the operators $\rho_{\sigma\pm}(k)$ to represent the central degrees of freedom of the theory. For a Hubbard interaction, this gives

$$\begin{aligned} H_U &= \frac{U}{2} \sum_\sigma \int dq\, \rho_\sigma(q) \rho_{\bar{\sigma}}(-q) \\ &= \frac{U}{2} \sum_\sigma \int dq\, \left[\rho_{\sigma+}(q) + \rho_{\sigma-}(q) \right] \left[\rho_{\bar{\sigma}+}(-q) + \rho_{\bar{\sigma}-}(-q) \right] \\ &= \frac{U}{2} \sum_{\sigma, s=\pm} \int dq\, \left[\rho_{\sigma s}(q) \rho_{\bar{\sigma} s}(-q) + \rho_{\sigma s}(q) \rho_{\bar{\sigma}\bar{s}}(-q) \right] \end{aligned} \tag{15}$$

This is not general; this is just the case for Hubbard interaction. Moreover, due to the book "Condensed Matter Field Theory" by Ben and Simons, for a general interaction, the above equation can be written in the form of:

$$H_{\text{int}} = \frac{1}{2L} \sum_{\sigma,s=\pm} \int dq \; [g_4 \rho_{\sigma s}(q)\rho_{\bar{\sigma} s}(-q) + g_2 \rho_{\sigma s}(q)\rho_{\bar{\sigma}\bar{s}}(-q)] \qquad (16)$$

where g_4 and g_2 gives two coefficients which are not required to be equal. The commutation relations for the momentum-space density operators are,

$$\begin{aligned} & [\rho_{\sigma s}(k), \rho_{\sigma' s'}(k')] \\ = \;& \delta_{\sigma\sigma'}\delta_{ss'} \int dq \; (\psi^+(q+k+k')\psi(q) - \psi^+(q+k')\psi(q-k)) \end{aligned} \qquad (17)$$

which is different from the book of "Condensed Matter Field Theory" on Page.70 on a minus sign. To make further progress, we must resort to a (not very restrictive) approximation. Ultimately we will want to compute some observables involving quantum averages taken on the ground state of the theory, $\langle 0|\hat{A}|0\rangle$. This approximation seems reasonable, since what we are interested is always the system with large numbers of particles' collective motion, of order $\sim 10^{23}$, the system, at low temperature, is almost at its ground state, except for very few of them excited. To simplify the structure of the theory, we may thus replace the right-hand side of the relation by its ground state expectation value:

$$\begin{aligned} & [\rho_{\sigma s}(k), \rho_{\sigma' s'}(k')] \\ \approx \;& \frac{L}{2\pi}\delta_{\sigma\sigma'}\delta_{ss'} \int dq \, \langle 0|\psi_s^+(q+k+k')\psi_s(q) - \psi_s^+(q+k')\psi_s(q-k)|0\rangle \\ = \;& \frac{L}{2\pi}\delta_{\sigma\sigma'}\delta_{ss'}\delta_{k,-k'} \int dq \, \langle 0|\rho_{qs} - \rho_{q-k,s}|0\rangle \end{aligned} \qquad (18)$$

At the first glance, it seems the r.h.s. of Eq.(17) vanishes. Due to a simple shift of the momentum index, $\sum_k \langle 0|\rho_q|0\rangle = \sum_k \langle 0|\rho_{q-k}|0\rangle$. But, this is too trivial result as we are not expecting. The summation on momentum, k, tells us that the two terms contributing to the sum cancel. However, this argument is certainly too trivial. Consider the upper limit and lower limit of the momentum summation, we find this argument It ignores the fact that our summation is limited by a cut-off momentum Λ. Since the shift $q \to q-k$ changes the cut-off, the argument of course forget this subtlety above. Keep in mind that in the ground state, all states with momentum $k < 0$ are occupied (it is the Fermi sea) while all states with $k > 0$ are empty. This implies that,

$$\begin{aligned}
\int dq \, \langle 0|\rho_{q+} - \rho_{q-k+}|0\rangle &= \frac{L}{2\pi}\left(\int_{-\Lambda}^{0} + \int_{0}^{q} + \int_{q}^{\Lambda}\right) dq \, \langle 0|\rho_{q+} - \rho_{q-k+}|0\rangle \\
&= \frac{L}{2\pi}\int_{0}^{q} dq \, \langle 0|\rho_{q+} - \rho_{q-k+}|0\rangle = -\frac{qL}{2\pi} \\
\int dq \, \langle 0|\rho_{q-} - \rho_{q-k-}|0\rangle &= \frac{qL}{2\pi} \qquad (19)
\end{aligned}$$

Therefore Eq.(17) can get into this result,

$$[\rho_{\sigma s}(k), \rho_{\sigma' s'}(k')] = -\delta_{\sigma\sigma'}\delta_{ss'}\delta_{k,-k'}\frac{qLs}{2\pi} \qquad (20)$$

Unfortunately the commutation relation of the density function depends on the momentum q. If we can absorb this parameter into the operators, we can of course obtain the bosonic commutation relation. Let us define,

$$\begin{aligned}
b_{\sigma q} &= \sqrt{\frac{2\pi}{qL}}\rho_{\sigma -}(q), & b^{+}_{\sigma q} &= \sqrt{\frac{2\pi}{qL}}\rho_{\sigma -}(-q) \\
b_{\sigma -q} &= \sqrt{\frac{2\pi}{qL}}\rho_{\sigma +}(-q), & b^{+}_{\sigma -q} &= \sqrt{\frac{2\pi}{qL}}\rho_{\sigma +}(q) \qquad (21)
\end{aligned}$$

Therefore, the interaction term can be expressed as,

$$U_{\text{int}} = \frac{1}{2\pi}\sum_{\sigma}\sum_{q>0} q \begin{pmatrix} b_{\sigma q} & b^{+}_{\sigma -q} \end{pmatrix} \begin{pmatrix} g_4 & g_2 \\ g_2 & g_4 \end{pmatrix} \begin{pmatrix} b^{+}_{\sigma q} \\ b_{\sigma -q} \end{pmatrix} \qquad (22)$$

This is a good-looking interaction, however, since the free part of the Hamiltonian is not written in terms of b operators, this problem is still unsolvable. However, when we recall the idea of the equations of motion for Heisenberg representation, i.e., $i\partial_0 \hat{A} = [\hat{A}, \hat{H}]$, we plug in $\hat{A} = \rho_{\sigma s}(q)$, what we can find is,

$$\begin{aligned}
&[H_f, \rho_{\sigma s}(q)] \\
= &\left[\hbar v_F \int dp \sum_{\sigma' s'} s' p \psi^{+}_{\sigma' s'}(p)\psi_{\sigma' s'}(p), \int dk \, \psi^{+}_{\sigma s}(k+q)\psi_{\sigma s}(k)\right] \qquad (23)
\end{aligned}$$

Using the indentity below,

$$\begin{aligned} & p\psi_p^+\psi_p\psi_{k+q}^+\psi_k - p\psi_{k+q}^+\psi_k\psi_p^+\psi_p \\ = \quad & (k+q)\psi_{k+q}^+\psi_k\delta_{k+q-p} - k\psi_{k+q}^+\psi_k\delta_{k-p} \end{aligned} \tag{24}$$

and plug this back into Eq.(22), we obtain the important relationship,

$$[H_f, \rho_{\sigma s}(q)] = sqv_F\rho_{\sigma s}(q) \tag{25}$$

This is an exciting result, recall Eq.(19), the intuition tells us that the free Hamiltonian can be written in operators of $\rho_{\sigma s}(q)\rho_{\sigma s}(-q)$, because one operator contraction with ρ gives a constant, to obtain the other operator in Eq.(24), we just need to add one more operator, to obtain quadratic operators representing the free Hamiltonian:

$$H_f = \frac{v_F}{L}\sum_{s\sigma}\int dq\,\rho_{\sigma s}(q)\rho_{\sigma s}(-q) \tag{26}$$

Now its time to take every term into account. Combining both Eq.(21) and Eq.(25), we obtain the Hamiltonian in the matrix form as below:

$$H = \sum_\sigma \int_0^\infty q \begin{pmatrix} b_{\sigma q} & b_{\sigma -q}^+ \end{pmatrix} \begin{pmatrix} \frac{g_4}{2\pi} + v_F & \frac{g_2}{2\pi} \\ \frac{g_2}{2\pi} & \frac{g_4}{2\pi} + v_F \end{pmatrix} \begin{pmatrix} b_{\sigma q}^+ \\ b_{\sigma -q} \end{pmatrix} dq \tag{27}$$

This is the process of Bosonization of the Luttinger Liquid. Recall the interaction term with the form of Hubbard interaction, the difference from here is just to let $g_4 = g_2$.

Diagonalizing this Hamiltonian Eq.(26), we can find the eigenvalues, i.e., the density operator's velocity,

$$v_\rho = \frac{1}{2\pi}\left[(2\pi v_F + g_4)^2 - g_2^2\right]^{1/2} \tag{28}$$

This is a free theory without spin interaction fermions which was developed in one dimension making use of the bosonization process. Eq.(27) showed that the low-energy degrees of freedom were described by hydrodynamic charge (i.e. density) fluctuations that propagated with a linear dispersion. This is the so-called Charge Density Wave's wave velocity (CDW), because except for a scaling factor of $\frac{2\pi}{qL}$, ρ operators (density operators) are equivalant to the b operator in the final Hamiltonian.

However, we still need to consider the spin-coupling terms, a general Hamiltonian can be written as: (with $q > 0$)

$$H = \sum_{\alpha sq} qv_F b_{\alpha sq}^+ b_{\alpha sq} + \sum_{\alpha\alpha' sq} q\left[\frac{g_2}{2\pi}\left(b_{\alpha sq}^+ b_{\alpha'\bar{s}q}^+ + h.c.\right) + \frac{g_2}{2\pi} b_{\alpha sq}^+ b_{\alpha' sq}\right] \tag{29}$$

Let us introduce operators to create charge ρ and spin σ fluctuations,

$$\begin{aligned} b_{sq\rho} &= \frac{1}{\sqrt{2}}\left(b_{sq\uparrow}+b_{sq\downarrow}\right) \\ b_{sq\sigma} &= \frac{1}{\sqrt{2}}\left(b_{sq\uparrow}-b_{sq\downarrow}\right) \end{aligned} \tag{30}$$

combining Eq.(31), Eq.(32) and Eq.(33), we obtain the new Hamiltonian after the transformation that,

$$\begin{aligned} H &= \sum_{q>0,s} q\left[v_F b^+_{sq\rho}b_{sq\rho} + \frac{g_2}{\pi}\left(b^+_{sq\rho}b^+_{\bar{s}q\rho}+h.c.\right) + \frac{g_4}{\pi}b^+_{sq\rho}b_{sq\rho}\right] \\ &+ \sum_{q>0,s} q\left(v_F b^+_{sq\sigma}b_{sq\sigma}\right) \end{aligned} \tag{31}$$

Now, we have successfully seperate this Hamiltonian into two independent parts, one with the solved charge density Hamiltonian, the other with the free spin density Hamiltonian. Rewrite this in terms of matrices,

$$\begin{aligned} H &= \int_0^\infty q\begin{pmatrix} b_{\rho q} & b^+_{\rho q-}\end{pmatrix}\begin{pmatrix} \frac{g_4}{\pi}+v_F & \frac{g_2}{\pi} \\ \frac{g_2}{\pi} & \frac{g_4}{\pi}+v_F\end{pmatrix}\begin{pmatrix} b^+_{\rho q} \\ b_{\rho q-}\end{pmatrix}\frac{dq}{2\pi} \\ &+ \int_0^\infty q\begin{pmatrix} b_{\sigma q} & b^+_{\sigma q-}\end{pmatrix}\begin{pmatrix} v_F & 0 \\ 0 & v_F\end{pmatrix}\begin{pmatrix} b^+_{\sigma q} \\ b_{\sigma q-}\end{pmatrix}\frac{dq}{2\pi} \end{aligned} \tag{32}$$

Therefore we can safely reach the goal of the charge density wave (CDW) velocity and spin density wave (SDW) velocity seperate. This violates the single-pole assumption at the origin of the Fermi liquid. Anticipating the meaning of the two poles is clear: charge-spin separation.

$$v_c = \left[\left(v_F+\frac{g_4}{\pi}\right)^2-\left(\frac{g_2}{\pi}\right)^2\right]^{1/2}, \quad v_s = v_F \tag{33}$$

Despite the nicety of the electron, and the apparent ubiquity of this phenomenon, the observation of spincharge separation in one-dimensional conductors has presented a significant challenge to experimentalists. The reason is that, the completion of an electrical circuit necessarily requires contact of the quantum wire with bulk leads. The leads involve a reservoir of electrons with conventional Fermi-liquid character. Electrical transport requires the recombination of the collective

charge (holon) and spin (spinon) degrees of freedom at the contact to reconstitute physical electrons. It is an exasperating fact that this reconstitution of the physical electron masks the character of spin-charge separation. Instead, the phenomenon of spincharge separation has been inferred indirectly through spectroscopic techniques.
These are the two experimental pictures I find from "Probing Spin-Charge Separation in a Tomonaga-Luttinger Liquid" by Y. Jompol, et al.

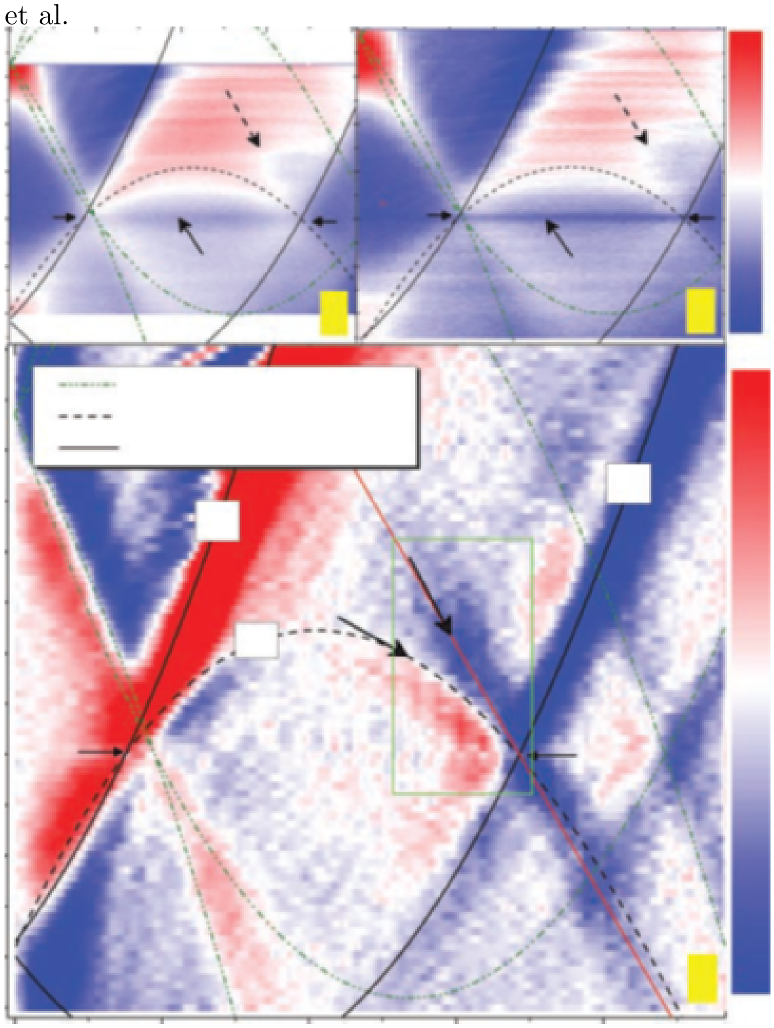

Fig.1. (A and B) Color-scale plots of G versus V_{dc} and B at lattice temperatures of 1K and 40mK. Black lines (solid and dashed) indicate the locations of singularities predicted by the noninteracting modelfor tunneling between the wires and the 2DEG, whereas the green dash-

dotted lines indicate the locations of the singularities associated with the parasitic 2D-2D tunneling. There is an additional abrupt decrease in G along the line indicated. In addition, G is suppressed at zero bias, labeled ZBA; this is another sign of interactions. (C) dG/dB (device A, for $V_{wg} = 0.60V$). The noninteracting parabolae are shown as in (A) and labeled 1D or 2D to indicate which dispersion is being probed. The straight red line indicates the locus of the abrupt change in G indicated in (A) and (B) and is a factor of 1.4 steeper than the 1D parabola at $V_{dc} = 0$. This feature clearly moves away from the 1D parabola. They identify it with the TLL charge excitation (holon), whereas the 1D parabola tracks the spin excitation (spinon).

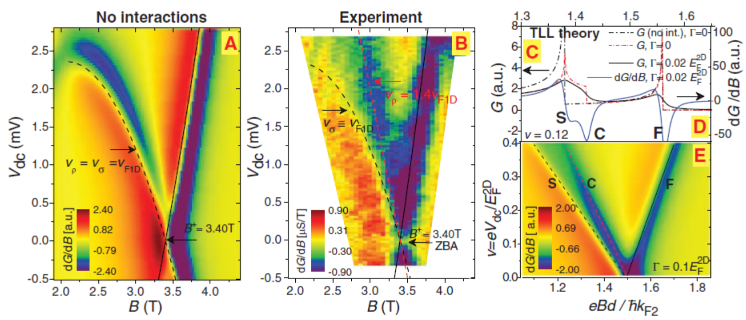

Fig.2. Comparison of dG/dB for experiment and theory. (A) For noninteracting electrons, all features track the noninteracting parabolae. (B) dG/dB measured at high resolution while sweeping B, for device B. The red line marks a feature that does not track the noninteracting parabolae and is absent in (A). Calculation of $G(C)$ and dG/dB (D) for noninteracting electrons. The dimensionless bias $v = eV_{dc}/E_F$, 2D=0.12; G is indicated for each curve. The spinon velocity $v_s = v_F$ in 1D and the chosen holon velocity $v_\rho = 1.4v_F$ in 1D. Spin and charge excitations are labeled S and C, respectively. F labels the noninteracting 2D dispersion curve. (E) dG/dB as a function of B and v, showing the same charge feature (C) as in the experiment (B).

Finally, the Hubbard Model is still not fully solved. Actually the mathematical tools above are insufficient for our problem. We need Jordan-Wigner Transformation for the one-dimensional Fermi system, to obatin the kink function. The final result for Hubbard Interaction turns out to be Sine-Gordan Model. Thus, for the charge density's Hamiltoinan, we obtain a free Hamiltonian while for the spin denfity's Hamiltonian, we get a Sine Gordan Model. The free part of spin density Hamiltonian gives a group velocity which is still different from that of charge density group velocity. I will go through detail deductions for the part in this winter holiday.

3 Conclusion

Due to the breaking down of Fermi Liquid Theory in 3-Dimensional Fermi system, we found another way to solve this problem, called the 1-D Fermi system bosonization. On the other hand, we can find the Jordan-Wigner Transformation has the similar physics idea with that of bosonization, by introducing kink function. By choosing the fermions' density function as the new set of creation-annihilation operators, we finally diagonalize this Hamiltonian, and find two set of operators – charge density operators and spin density operators have different group velocities. The charge density wave (CDW) and spin density wave (SDW) are the emergent state of 1-Dimensional Fermion system. Since the experiments observed spin-charge separation in the two pictures above due to the difference on the group velocity, we conclude that this emergent state exists in 1-Dimensional system.

References

[1] "One-dimensional Fermi liquids" by Johannes Voit (1994)

[2] "Transport in a one-dimensional Luttinger liquid" by Matthew P. A. Fisher and Leonid I. Glazman (2008)

[3] "Lectures on Bosonization" by C.L. Kane (2005)

[4] "Quantum Physics in One Dimension" by Thierry Giamarchi (2003)

[5] "Luttinger liquid theory of one-dimensional quantum fluids. Properties of the Luttinger model and their extension to the general 1D interacting spinless Fermi gas" by F D M Haldane (1981)

[6] "Condensed Matter Field Theory" by Ben Simons (2010)

[7] "Advance Solid State Physics" by Phillip Phillips (2011)