Kuramoto Oscillators

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Abstract

The Kuramoto model for systems of oscillators, a first-order system of differential equations used to study systems of phase oscillators, is a useful tool for the study of synchronization. This model occupies an essential niche between triviality and reality, being complex enough to have interesting features and yet admitting solution. In this paper we discuss the formulation of the model for a finite and infinite number of oscillators, its applications to physics and neuroscience, and some recent mathematical developments.

1 Introduction

1.1 Synchronization

The phenomenon of synchronization pervades everyday experience. Some examples are hardly surprising, like the synchronization of the front and back wheels of a bicycle, as they are highly coupled. Others, like the momentary synchronization of turn signals in traffic, which aren't really coupled at all, don't really merit investigation. There are examples between these extremes, however, which are of surprising complexity and beauty. These examples include the synchronization of fireflies, to circadian rhythms observed in animals, and even the subtle synchronization of the heartbeat to music. [3][7] The fundamental feature of all of these examples is the presence of several objects which can each said to be oscillating, and the phenomenon of some of the objects influencing the oscillations of others. The problem with a naive viewpoint on this subject is that every situation could potentially merit its own unique description, which would entail a unique analysis

[4]. While this is productive, it misses the larger point, focusing on differences rather than similarities, and it does not strive to address the seeming universality of the phenomenon. In light of this, it is promising to study a model which exhibits all the basic features of a system in sync, keeps enough flexibility to describe a wide range of situations, and yet is still tractable from an analytical perspective.

1.2 A Model

One model which strikes a compelling balance of generality and solvability was proposed by Kuramoto in 1975, and has the following form:

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} K_{ij} f(\theta_j - \theta_i) \qquad i = 1...N$$

To explain the notation a little, each oscillator θ_i has a preferred frequency ω_i, but is biased away from this by its interaction with other oscillators. This interaction f is periodic in the difference of the phases of the oscillators concerned, and is zero if the phases are identical. This leads one to believe that f is probably a polynomial or power series in terms of $\sin(\theta_j - \theta_i)$ which it can be, but for the sake of simplicity it's typically taken to be only the lowest order such term.[3][6]

2 Methods

In considering a large number of oscillators, it is wise to change the coupling somewhat. First, all of the weights are taken to be equal. This has two effects, one is that the graph of interactions of the oscillators is complete (in the sense of graphs), the second is that $\frac{K}{N}$ can be used to regulate the strength of coupling in the limit as $N \to \infty$. The second modification is that the oscillators should not interact with each other oscillator individually, but collectively through their average phase $\Psi(t)$. The resulting differential equation is

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sin(\Psi - \theta_i) \qquad i = 1...N$$

Using this notation, it's also possible to define an order parameter for the system. This parameter, r, ranges from 0 to 1 and measures the degree to which the system is synchronized. It is defined as

$$re^{i\Psi} = \frac{1}{N}\sum_{i=1}^{N} e^{i\theta_i}$$

As it happens, it's possible to simulate a finite number of these oscillators (even a large number by some accounts), but analysis becomes progressively more challenging. The approach taken with the Kuramoto oscillator is the usual one, that of taking the limit $N \to \infty$ with $\frac{K}{N} = C$, a constant. If we do this, we can define the continuum analogue of the order parameter above, and discuss the oscillator behavior as a field quantity. This does involve a transition in notation and technique, switching to distribution functions, and solutions to Fokker-Planck equations as primary investigatory techniques.[6]

In this setup, we can also discuss bulk properties of a system of oscillators in terms of their phases as given by the order parameter. As the system is set up to be very general, one would expect that the correct choices of the coupling parameters could give rise to many kinds of behavior, and indeed there are examples where the system can undergo transitions of phase of both the discontinuous (first-order) and continuous (second-order) types. [3] As a side-note, this order parameter has its advantages and disadvantages. It does determine the amount of order in some sense. However, if the system were (for example) to exhibit two populations with different phases, say $\theta_1 = -\theta_2$, you could cook up a situation wherein the system can be split into halves, each of which is perfectly synchronized, but the overall synchronization is 0. [3]. It should also be noted that the question of stability is an extremely hard one to answer, and the stability of distributions of oscillators wasn't proven for some time. Beyond just this question, in a case with an infinite number of oscillators, the exact distribution of initial conditions matters a great deal. If there are no peaks, or one peak in the distribution (the simplest possible examples), the analysis is still extremely technical. [6]

3 Applications

3.1 Chemistry

The Kuramoto model is applicable to many problems in certain branches of science. Several of these will be discussed below. While these are instructive, none are as immediately fascinating (or visible) as that of the Belusov-

Zhabotinski reaction. This oscillating chemical reaction can be modeled with 2 linked oscillators which are perturbed by a random spatial imperfection. In addition to the figure, the reader may find that several pleasing examples are readily viewable on youtube.

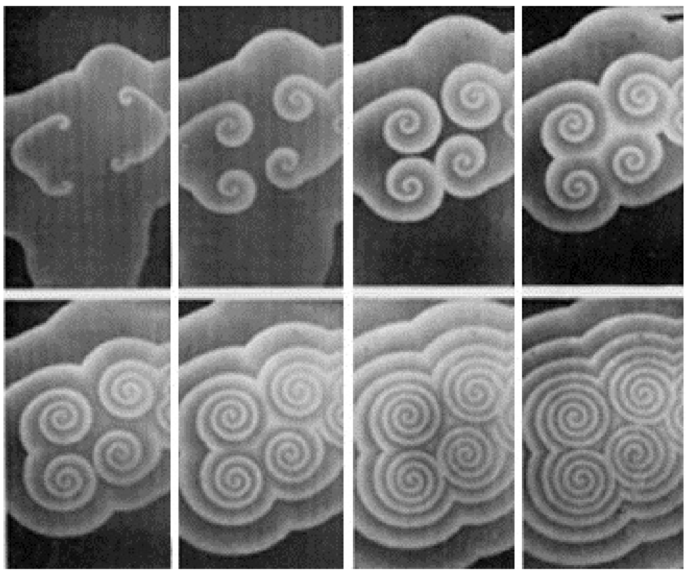

Figure 1: Development and propagation of spiral waves. Taken from Zhabotinsky and Zaikin, 1971

3.2 Neuroscience

Neuroscience can be done mathematically with several kinds of models. For some processes, the Kuramoto model is appropriate. These processes include those taking place in certain areas of the visual cortex. This type of application introduces many technical obstacles to mean-field and simulation calculation both, as the connection is no longer all-to-all. Instead, the overall network is sparsely connected, but locally it can be very densely connected. The response of the network can, in this case, be measured in terms of the number of oscillators having a specific phase at or about a specific time. [3]

Aside from actively modeling the brain, it's possible to model different methods by which neurons could hypothetically learn information. One such example is Hebbian learning, is taken to model learning and memory formation in the Hippocampus. This is accomplished by taking a system of

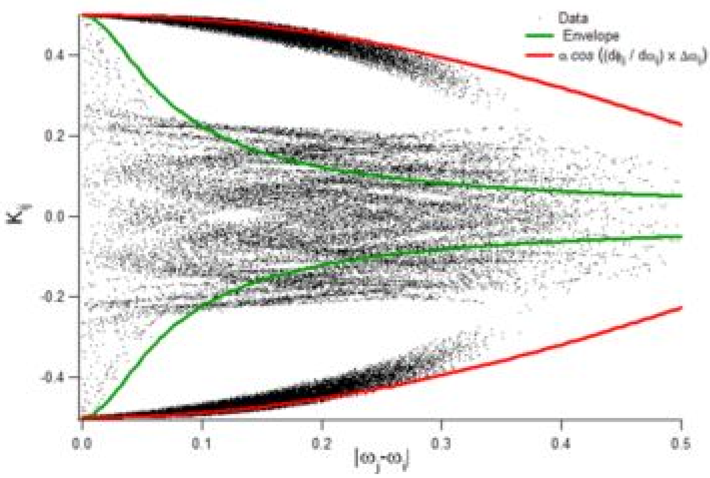

Figure 2: Oscillators in this simulation form two synchronized clusters in densely populated bands, while the uncoupled oscillators remain predominantly in the envelope in the middle. Taken from [5]

coupled phase oscillators, considering them to fire if their phase crosses a certain point, and reinforcing or diminishing the connections between neurons if they fire close to each other (diminishing would happen in considering directed connectivity, and reducing the value of a connection of the connection is from a to b and b fires before a does). The amount of change is loosely a "learning rate" and varying this rate can cause different structures of connectivity to emerge from a network. This example, as influenced by the learning rate, is subject to phases in much the same way as the original Kuramoto system.[5]

3.3 Physics

The Kuramoto oscillator model has also found many applications in physics, some of the most notable ones being the propagation of charge-density waves in quasi-one-dimensional metals and semiconductors. More specifically, the depinning transition can be studied by the addition of a term to the oscillator model which introduces a random pinning angle α, and considering the reaction to an applied electric field E, these have the form:

$$\dot{\theta}_i = E - h\sin(\theta_i - \alpha_i) + \sum_{j=1}^{N} K_{ij} f(\theta_j - \theta_i) \qquad i = 1...N$$

Another prominent application concerns Josephson junctions, which can be used as voltage-to-frequency transducers. To achieve a large output power, the junctions can be combined, just so long as they are synchronized properly. In fact, junctions connected in series exhibit the all-to-all coupling typical of the original model. The same is also true of lasers, which have to be phase-synchronized with one another in order to maximize power output.[3]

4 Recent Developments

There are, of course, more technical questions to be asked. First of all, if a system has a synchronized state, and is started out of that state, how long does it take to get there? If the system is perturbed by noise, is there still synchronization on average?

In answer to the first question, it depends. For certain ranges of coupling constants, synchronization is never achieved (this is expected, especially for very small coupling). There is a critical value of the coupling constant, which can be defined up to some technical points, after which synchronization happens. Furthermore, if the oscillators are going to synchronize, the will do so exponentially quickly. [1]

In answer to the latter question, if a system of oscillators manages to synchronize and is subject to noise below a certain threshold, the system spends the vast majority of its time near the synchronized state. However, the system will occasionally undergo a large deviation, wherein one or more of the oscillators slips, and the system then resettles into a new stable state. This transit time between stable states is extremely short, and under appropriate assumptions, does not increase if the number of oscillators is increased.[2]

5 Concluding Remarks

Synchronization is a growing area of research–it will likely remain popular for some time as this phenomenon so compelling and readily observable in the world. To study this, the Kuramoto model model has many advantages;

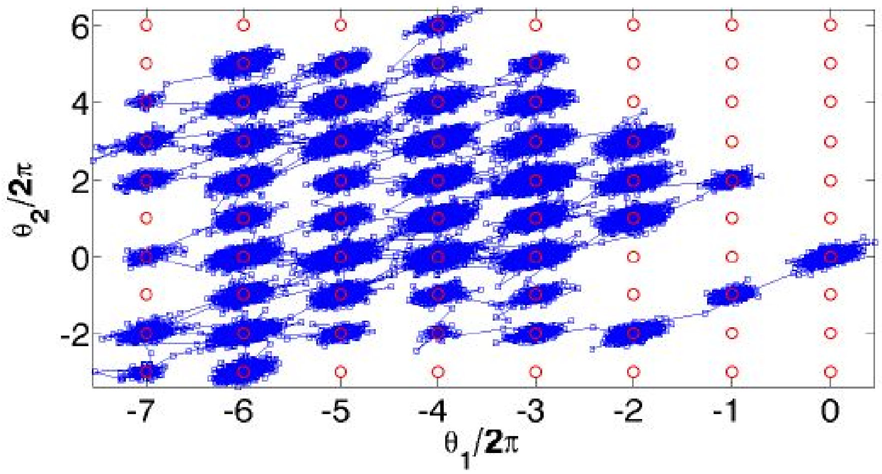

Figure 3: A simulation of transitions of a stochastic Kuramoto system between stable states (the exact stable states are given by the small circles). Taken from [2]

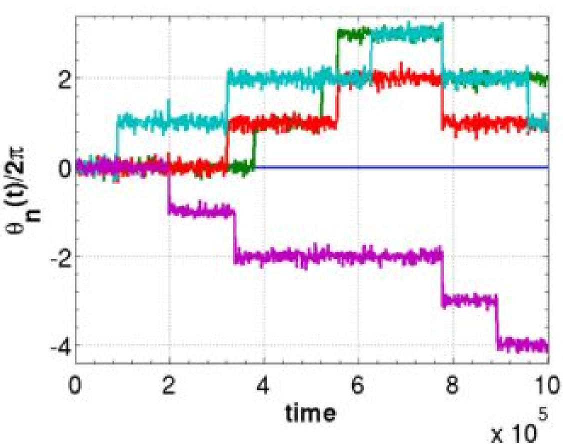

Figure 4: A simulation of 5 oscillators illustrating the relative times spent in stable configurations and transitioning between them. Taken from [2]

it is used throughout many fields because of it's tractability, and ease of simulation. It remains an active area of mathematical research, harboring unanswered questions including the effects of various network topologies on synchronization and stability. And, in the case with an infinite number of oscillators, the question of the behavior of systems with multimodal distributions is not yet solved completely.

References

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