Cosmic Strings and Topological Defects

Jiawen Liu

December 9, 2012

Abstract

In this review article, we point out spontaneous symmetry breaking is closely related to the emergence of the topological defects. Basic definitions and dynamics of the domain walls are introduced in the second section. Finally, we worked out an example of the spherical domain wall by using semi-classical approach.

1 Introduction

Cosmic strings are one dimensional topological defects which may arise from a symmetry breaking phase transition of the early universe. Consider the geometry of the early universe is a non-connected vacuum manifold, $\mathcal{M}$. Cosmic strings are characterized by non-trivial fundamental group of the manifold, $\mathcal{M}(\pi_1(M) \neq I)$, which is an analog of the topological defect of two dimensional vortex. The manifold, $\mathcal{M}$ is not contractible which means the singularity corresponding to a hole cannot be removed by continuous deformation. There are other topological defects in the curved spacetime such as monopoles, domain walls, and textures.

This expository article is mainly focused on the basics of the cosmic strings and topological defects. We basically give a short account of the theory based on the book by Vilenkin and Shellard. The article is organized as follows. First, introduce the origin of the topological defects in the universe. In the book, they argue that the symmetry of Higgs fields (scalar fields) break because of the non-zero expectation value of the ground state. Hence, we can heuristically determine that the manifold has some non-trivial topological properties. Second, we explain the concept of domain walls and their non-trivial topological properties. Finally, we will work out an example of the defects in de Sitter space (Minkowski spacetime), which is closely related to the ideas of domain walls and the ideas of inflation in the early universe. We do not guarantee the correctness of this article, because the reference is rather outdated and there are new discoveries that may disprove the arguments addressed in this article. However, from my personal viewpoint, the idea of spontaneous symmetry breaking associated with cosmology is very interesting, because it perhaps addresses the fundamental question of the true ground state of the early universe.

2 Spontaneous Symmetry Breaking

The ideas of spontaneous symmetry breaking usually associated with second order (continuous) phase transition originally come from condensed matter physics. The phenomena of superconductivity can also be understood via spontaneous symmetry breaking, where the photon becomes massive due to global $U(1)$ gauge symmetry breaking. Hence, spontaneous symmetry breaking is closely related to the macroscopic, non-localized phenomena, namely

BEC (Boson Einstein Condensation). In high energy physics, gauge bosons become massive due to Higgs Mechanism. Eq. (1) features the idea of spontaneous symmetry breaking.

$$\mathcal{L} = (\partial_\mu \bar{\phi})(\partial^\mu \phi) - V(\phi) \tag{1}$$

where $V(\phi) = \frac{1}{4}\lambda(\bar{\phi}\phi - \eta^2)^2 (\lambda, \eta > 0)$.

After some algebra, we can verify the Lagrangian is invariant under global gauge transformation, where $\phi(x) \to e^{i\alpha}\phi(x)$. To find the ground state, we take derivative of the potential with respect to ϕ, $\partial V(\phi)/\partial\phi = 0$. So the corresponding field is described by a circle, $|\phi| = \eta$. This result leads to an interesting point. We will get a non-zero expectation value that has a phase factor $e^{i\theta}$. Furthermore, the expectation value featured by the ground state is not invariant under global gauge transformation. Hence, we can deduce that the symmetry of vacua is spontaneously broken.

In Problem 8-1, we further study the broken symmetry of vacua by variational analysis of the abelian-Higgs Model.

$$\mathcal{L} = \bar{\mathcal{D}}_\mu \bar{\phi} \mathcal{D}^\mu \phi - V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{2}$$

where ϕ is a complex scalar field and $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative. Choose the phase of ϕ to be zero when the magnetic field is absent, $\phi_0 = \eta$. We can proceed to expand ϕ up to second order around ϕ_0 in terms of the real and the imaginary parts of the fluctuations, ϕ_1 and ϕ_2, respectively.

$$\phi = \mu + (\phi_1 + i\phi_2) \tag{3}$$

where I have been a little bit sloppy here without taking account of the normalization factor of the wave function. Now, under the $U(1)$ local gauge transformation,

$$\phi(x) \to e^{i\alpha(x)}\phi(x), \;\; A_\mu(x) \to A_\mu(x) + e^{-1}\partial_\mu \alpha(x) \tag{4}$$

we find that ϕ_2 goes away under the prescription of the broken-symmetry of the vacua, which means one degree of freedom of the scalar field is absorbed into the massive photon field, A_μ. This model, as shown in the HW problem, corresponds to the Ginzburg-Landau model of a superconductor, where ϕ becomes the Cooper pair wave function.

It is natural to ask the following questions. Since the topology of the manifold, M where ϕ lives, has topological defects due to the symmetry breaking (second order phase transition); what are those unbroken symmetry and how to describe them? [1] In the next section, we will consider the simplest example of the topological defects, ϕ^4 kinks.

3 Domain Walls

Having introduced the idea of spontaneous symmetry breaking, we are now at the position of introducing the topological defects. Consider the cases of a Goldstone model,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi) \tag{5}$$

where ϕ is the scalar field and the potential is of the form, $V(\phi) = \frac{\lambda}{4}(\phi^2 - \eta^2)^2$. By setting the Lagrangian to be zero[2], we have a non-trivial analytic solution.

$$\phi(x) = \eta \tanh((\lambda/2)^{1/2}\eta x) \tag{6}$$

The calculation is done as follows,

$$\frac{1}{2}(\frac{\partial \phi}{\partial x})^2 = \frac{\lambda}{4}(\phi^2 - \eta^2)^2 \tag{7}$$

$$\frac{\partial \phi}{\partial x} = -\sqrt{\frac{\lambda}{2}}(\phi^2 - \eta^2) \tag{8}$$

$$-\frac{d\phi}{(\sqrt{\frac{\lambda}{2}})(\phi^2 - \eta^2)} = dx \tag{9}$$

Take integral on both sides and take the initial point to be at $x = 0$. We can recover Eq. (6). The other trivial solutions to the Equation $\mathcal{L} = 0$ is $\phi = \pm\eta$. The additional "vacuum" state is a localiezed "kink" centered about $x = 0$. $\phi(x) = \eta$ and $-\eta$ as x goes to $+\infty$ and $-\infty$, respectively. If the kink is centered about $x = x_0$, Eq. (6) becomes $\phi(x,t) = \eta \tanh((\lambda/2)^{1/2}\eta(x - vt))$.[3]

[1] Topological defects cannot be removed by continuously deforming the shape that features the field. In this case, the topological defects cannot deform into the vacuum state. In the language of group theory, a global symmetry described by a group, G, is broken to a sub-group of of G such that $G \subseteq H$ and $\mathcal{M} = G/H$, where M is the coset of H in G.

[2] the kinetic energy is equivalent to the potential energy

[3] Here we also consider the Lorentz invariance of the theory so that the solition solution can boost to any arbitrary velocity. The solution is valid for any non-relativistic velocity v.

To see how the topological defect comes about, we have the following reasoning. Note the analytical solition solution, Eq.(6) is time-dependent, which means the state is stable. Hence, the disconnected manifold, $\mathcal{M}$ has a non-trivial topological property. Moreover, it costs a large amount of energy to remove the topological defect. Or we may remove the defect by putting an anti-defect into the system, which carries the opposite topological orientations of the defect. We can also think of the topological orientations as charges. The charges are topological charges governed by the topological conservation laws. Without proof, I directly copy the expression of the topological current from the book,

$$j^{\mu} = \epsilon^{\mu\nu}\partial_{\nu}\phi \tag{10}$$

The above equation is derived from the topological conservation laws.

To better understand Eq. (10), we can relate the topological conservation laws to the Noether theorem, wher the charge $Q = \int dx j_0$ is conserved.[4] So the associated conserved charge is

$$N = \int dx j^0 = \phi|_{x=+\infty} - \phi|_{x=-\infty} \tag{11}$$

where the non-zero N implies that the ϕ^4 kink is stable. This example is the simplest case of the topological defects. There are some general cases which cannot be described by analytical solition solutions. We must present the higher dimensional topological defects in the context of topology.

3.1 Basic properties of the Domain walls

From the previous discussion, we know that Z_2 symmetry ($\phi \leftrightarrow -\phi$) is broken due the appearance of the ϕ^4 kink. Or we can say the appearance of domain walls is generally associated with a discrete symmetry breaking. Domain walls usually occur at the boundaries between two regions, $\phi = -\eta$ and $\phi = \eta$. We can also approximate the width of the wall with Eq. (6).

$$\delta \sim (\sqrt{\lambda}\eta)^{-1} \tag{12}$$

where we have used the fact $\lim_{x\to 0}\tanh(x) = x$ and locally the curvature radii of the wall is much greater its thickness.We can also estimate the surface energy density by using the vacuum energy at the center of the wall, $\rho \sim \lambda\eta^4$.[5]

[4] Note that the dimension of the space is 1.

[5] The result directly follows from the form of the potential, $V(\phi) = \frac{\lambda}{4}(\phi^2 - \eta^2)^2$. And we can find the vacuum energy by setting ϕ to be zero.

Furthermore, we may conclude that the surface energy density is sufficiently large unless the symmetry scale η is very small. That's why the domain walls are very important to the homogeneity of the universe. In next section, we will give a detailed calculation of the dynamics of the domain walls.

3.2 Domain wall dynamics

First of all, we need to describe the setup of this calculation. In most cases, the thickness of the domain wall can be neglected and the wall can be treated as a thin surface. The spacetime can be described by a three dimensional worldsheet.

$$x^\mu = x^\mu(\zeta^a), a = 0, 1, 2 \tag{13}$$

The action describing the wall is

$$S = \int d^4x\sqrt{-g}\,(\frac{1}{2}(\partial\phi)^2 - V(\phi)) \tag{14}$$

Now we need to write down the metric of the worldsheet. For convenience, we use the coordinates ζ^a and z, where ζ represents the surface and z is distance normal to the surface. The metric in these coordinates is of the form

$$ds^2 = \gamma_{ab}d\zeta^a d\zeta^b - dz^2 \tag{15}$$

where $\gamma_{ab} = g_{\mu\nu}(\frac{dx^\mu}{d\zeta^a})(\frac{dx^\nu}{d\zeta^b})$. The spacetime volume element is

$$\sqrt{-g}d^4x = \sqrt{\gamma}d^3\zeta dz \tag{16}$$

where $\gamma = \det(\gamma_{ab}) = \exp(tr\ln(\gamma_{ab}))$. With the spacetime element, we can write the string action as

$$S = \int d^4x\sqrt{-g}\,\mathcal{L} \tag{17}$$

$$= -\sigma\int d^3\zeta\sqrt{\gamma} \tag{18}$$

where we have integrated out the z dependence in the last step and $\sigma = -\int \mathcal{L}dz$. Eq. (18) can be thought as the three-volume of the wall's worldsheet and σ can be interpreted as the mass per unit area of the wall. Variation of Eq. (17) with respect to the metric $g_{\mu\nu}$ gives rise to the energy momentum tensor.

$$T^{\mu\nu} = -2\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}} = \sigma \int \delta^(4)[x - x(\zeta)]\sqrt{\gamma}\gamma^{ab}x^{\mu}_{,a}x^{\nu}_{,b} \tag{19}$$

where we have used the relation $(\gamma_{ab})^{-1} = \gamma^{ab}$ and the definition of the $\det(\gamma_{ab})$.

We can also determine the equations of motion of the wall by varying the action with respect to $x^{\mu}(\zeta)$.

$$\Box x^{\mu} + \Gamma^{\mu}_{\nu\sigma}\gamma^{ab}x^{\nu}_{,a}x^{\sigma}_{,b} = 0 \tag{20}$$

where $\Box$ is the Beltrami differential $\frac{1}{\sqrt{\gamma}}\partial_a(\sqrt{\gamma}\gamma^{ab}\partial_b)$. We also have assumed the metric is preserved such that $\bigtriangledown_{\mu} g_{\mu\nu} = 0$ (Levi-civita connection). To further simplify the problem, we take $g_{\mu\nu}$ to be $\eta_{\mu\nu}$. So Eq. (20) can be reduced to

$$\partial_a(\sqrt{\gamma}\gamma^{ab}x^{\mu}_{,b}) = 0 \tag{21}$$

Under a choice of gauge,

$$\gamma_{01} = \gamma_{02} = 0, \gamma_{00} = \sqrt{\gamma} \tag{22}$$

This is actually the conformal gauge. So Eq. (21) becomes

$$\ddot{x} + \epsilon^{AC}\epsilon^{BD}\partial_A(x^{\mu}_{,D}x_{\nu,D}x^{\mu}_{B}) = 0 \tag{23}$$

where we have defined $\gamma^{AB} =^{[2]} \gamma^{-1}\epsilon^{AC}\epsilon^{BD}\gamma_{CD}$, $\gamma_{AB} = x^{\nu}_{,A}x_{\nu,B}$ and ${}^{(2)}\gamma = \gamma_{00} = \det \gamma_{AB}$. We can find a family of solutions $\mathbf{x}(\eta^1, \eta^2, t)$ by introducing an ansatz.

$$\mathbf{x}(\eta^1, \eta^2, t) = \mathbf{n}\zeta^2 + \mathbf{x}_{\perp}(\eta^1, t) \tag{24}$$

where $\mathbf{n}$ is an unit vector and $\mathbf{x}_{\perp}$ is on the plane that is perpendicular to $\mathbf{n}$. Substitute the ansatz into Eq. (23) and the gauge expressions.

$$\ddot{\mathbf{x}}_{\perp} - \mathbf{x}''_{\perp} = 0 \tag{25}$$

$$\dot{x}_{\perp} \cdot x'_{\perp} = 0 \tag{26}$$

$$\dot{x}^2_{\perp} + x'^2 = 1 \tag{27}$$

From the above expressions, we know that only the motion transverse to the wall is observable.

4 Inflation and Defects in de Sitter Space

Now several questions arise: how can we find domain walls in the universe and how to construct a model for such process? Cosmic inflation is a theorized scenario of the rapid expanding early universe. The expansion process is driven by the negative-pressure. The Big Bang model describes the evolution of the universe from 1/100 after initial explosion. The model is based an assumption that the universe is homogeneous and isotropic. Therefore, the present universe arises from a tiny initial region.At the early stage of the universe the "false" vacuum is thermalized and the universe expands from this point. People hope they can observe topological defects are formed after or near the end of the inflation, where the inflation is driven the vacuum energy.

4.1 Instanton Approach

In the book, the authors state that spherical domain walls can be continuously created during the inflation process. So, we may extract the pre-inflationary phase transitions by looking at those topological defects. Here, we present an example of spherical domain walls by using instanton approach[6]. The corresponding spacetime is described by a four-sphere, where the three-sphere represents the world surface of the wall. The four-sphere lies in a five-dimensional Euclidean space,

$$\vec{\zeta}^2 + \omega^2 + \tau^2 = H^{-2} \tag{28}$$

The instanton three sphere is

$$\vec{\zeta}^2 + \tau^2 = H^{-2} \tag{29}$$

$$\omega = 0 \tag{30}$$

The evolution of the domain wall can be determined by an analytical continuation to the Minkowski spacetime. Therefore, we have to connect Eq. (28) and Eq. (29) by the stereographic mapping method. The de-Sitter space can be realized as the usual heperboloid

$$\vec{\zeta}^2 + \omega^2 - \tau^2 = H^{-2} \tag{31}$$

[6]Instanton is a classical solution of the quantum field theory living in the Euclidean spacetime.

and the worldsheet surface is

$$\vec{\zeta}^2 = H^{-2} + \tau^2 \tag{32}$$

$$\omega = 0 \tag{33}$$

We have to write a metric describing how the evolving wall would be viewed by an observer in an inflationary universe.

$$ds^2 = dt^2 - e^{2Ht} d\mathbf{x}^2 \tag{34}$$

In order to write down the metric above, we have to parameterize the hyperboloid coordinates in terms of the coordinates in Eq. (34), where the metric in de-sitter space time is $ds^2 = d\tau^2 - d\omega^2 - d\vec{\zeta}^2$.

$$\tau = H^{-1}\sinh(Ht) + \frac{1}{2}H\mathbf{x}^2 e^{Ht} \tag{35}$$

$$\omega = H^{-1}\cosh(Ht) - \frac{1}{2}H\mathbf{x}^2 e^{Ht} \tag{36}$$

$$\vec{\zeta}^2 = \mathbf{x}e^{Ht} \tag{37}$$

Insert Eq. (35) – (37) into Eq. (33). We have the wall evolution equation,

$$\mathbf{x}^2 = H^{-2}(1 + e^{-2Ht}) \tag{38}$$

In the frame of the coordinates, Eq. (34), the wall is a sphere of radius

$$R = H^{-1}(e^{-2Ht} + 1)^{1/2} \tag{39}$$

The radius of the wall is H^{-1} as t goes to infinity. We have obtained an instanton solution, Eq. (38). In general, the expression can be replaced by

$$(\mathbf{x} - \mathbf{x_0})^2 = H^{-2}(1 + \exp[-2H(t - t_0)]) \tag{40}$$

where $\mathbf{x}_0$ and t_0 is dependent on the orientation of the three-sphere inside the four-sphere.

5 Conclusion

This review article covers the elementary aspects of the domain walls. However, the reference is rather dated. Matured readers should consult more advanced and recent cosmology texts.

References

[1] Vilenkin and Shellard *Cosmic Strings and other Topological Defects.* Cambridge Univ. Press, Cambridge, 1994.

[2] Goldenfeld, http://guava.physics.uiuc.edu/ nigel/courses/569/hw8.pdf. December, 8, 2012