Rare region effects in Ising models

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Abstract: The presence of quenched disorder gives rise to significantly different behavior in magnetic spin systems. This paper introduces the idea of rare regions, and analyses the effects of impurities and defects in several Ising systems. I will discuss the the Griffiths phases and their effect on thermodynamic variables and critical phenomena.

1 Introduction

The paper flow in the following manner: First I introduce Landau theory and state its limitations. Next, rare regions in the context of a dilute Ising ferromagnet and the Griffiths phase are presented. The rare regions are then classified more generally, their effects on thermodynamic variables, critical behavior are mentioned and examples for models in each classification are given. The limiting assumptions are stated as well as their effects on the dynamics of the system. Afterwards, I look more closely at the dilute Ising model: first treating the problem of percolation, discussing some properties and pointing out relevant literature. Next, the dilute Ising hamiltonian is presented and well as its phase diagram. Finally some experiments verifying the effects of the Griffiths phase are discussed and colcuding remarks are stated.

1.1 Landau Theory

Second order phase transitions can be characterized by an order parameter m, which is zero in the disordered phase, and non-zero in the ordered phase. In superconductors, for example, m is defined to be zero in the normal region at temperature above the critical temperature T_c, and is identical to unity for a perfect superconducting region at zero temperature. We can derive Landau theory if we start from the mean field equations from a magnet (Weiss) and a fluid (van der Waals). Its key observation is that near a critical point (where the order parameter becomes non-zero), the free energy is an analytic function of the order parameter m and can be expanded into a series [1]. The energy is of the form:

$$F = F_0 + rm^2 + vm^3 + um^4 + ... - hm$$

To get the value of the order parameter, we must minimize this free energy. If $v \neq 0$ the transition is discontinuous and thus is 1st order, these transitions wont be discussed. For most cases, however, $v = 0$ by symmetry. r is positive, and if it is very large we minimize the free energy by picking $m = 0$. Thus r is a measure of the distance from the critical point ($r \propto (T - Tc)$ for thermal transitions).

The problem with Landau theory is that it doesn't take into account fluctuations about the average value of the order parameter. It predicts the same mean-field critical behavior for all systems in all dimensions. This becomes a problem as fluctuations become of more importance, particularly, as the number of dimensions of our system is reduced. Critical dimensions can then be defined: d_c^+ is the upper critical dimension, where fluctuations are not important and mean field theory is valid, d_c^- is the lower critical dimension, where fluctuations are so strong that a phase transition cannot exist at any finite temperature. For the Ising model $d_c^+ = 4$ and $d_c^- = 1$ [2]. The region $d_c^+ < d < d_c^-$ has phase transitions at finite temperature, but critical behavior deviates from the one predicted by Landau theory.

2 Rare regions: Dimensionality matters!

Only the simplest type of disorder will be introduced in our model. That is quenched, or time-independent, disorder that leads to spatial variations of the coupling strength. This type of disorder only increases the tendency towards ordered phase withouth changing the qualitative behavior of the bulk phases. It is called weak disorder, random-T_c disorder, or random-mass disorder. In a real system, weak disorder is present in the form of impurities, vacancies, or grain boundaries.

One can model a rare region by randomly diluting a ferromagnet then labeling an aglomeration of cites that is *not* diute as the rare region Figure 1.

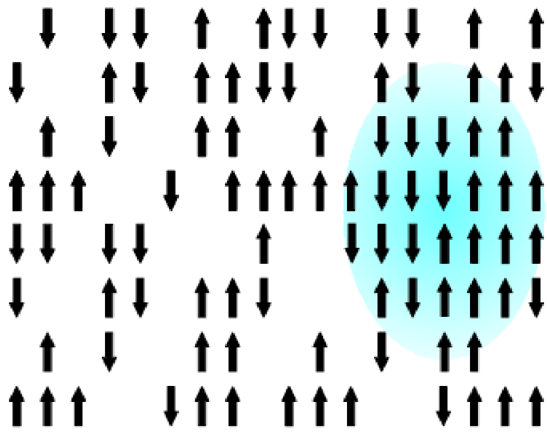

Figure 1: A randomly diluted magnet in $2d$ the shadowed area is the rare region, where we have an undiluted magnet. Taken from [2]. The probability of finding a large rare region decays exponentially as its volume is increased.

Intuitively, a dilute ferromagnet, with less nearest neighbor interactions, has a lower transition temperature T_c than the clean Ising system with transition temperature T_c^0. This turns out to be correct and was proven in [4]. Now, for the range $T_c < T < T_c^0$ in our system of clean-Ising-like impurities, our rare regions can begin ordering locally before the bulk! The range $T_c < T < T_c^0$ is called the Griffiths phase, and the rare regions give rise to a singularity in the free energy (called the Griffiths singularity) [5].

2.1 Classification of rare regions

In this section the rare regions are labeled as in [3] according to their dimensionality d_{RR} . To reveal the effects of the rare regions on the system, one must compare the exponentially decaying probability of finding a rare region $\propto e^{-V_{RR}}$ with increasing volume V_{RR}, to its increased contribution to the thermodynamic variables due again to its increasing volume. The comparison is controlled by the relation between the dimensionality of the rare region d_{RR} and the lower critical dimension of the system d_c^- [6]. There are three classifications:

Class A: $d_{RR} < d_c^-$ means the rare region cannot order by itself. Rare regions affect the thermodynamic properties insignificantly, and the critical behavior follows the same critical exponents. The point-like rare regions in the dilute magnet in Figure 1 cannot overcome this exponential decay in the probability, for example.

Class B: $d_{RR} = d_c^-$ the rare regions still cannot order by itself, but they are strong enough to overcome the exponential decay and change the behaviourof thermodynamic variables such as the suceptibility χ (which now depend exponentially with volume). Moreover, the system's behavior near the critical point is no longer conventional, having continuously varying exponents. An example of this case is the McCoy Wu model, which is a classical $2d$ Ising model with linear defects.

Class C: $d_{RR} > d_c^-$ the rare regions can order independently from the bulk and develop a static order parameter. The global phase transition of the system is *smeared* since different parts of the system are ordered for different parameters (like a different temperature). There is no scaling for the critical behavior. An example of this case is a $3d$ Ising model with planar defects.

These classifications hold as long as weak dissorder and short-range interactions are assumed. Moreover, the effects of rare regions interacting among themselves are neglected since their concentration is exponentially small. If this were not the case, a more careful tratment is neccesary. In some cases smeared transitions may occur even if the rare regions cannot order [7].

Another crucial point to note is that I have discussed only classical phase transitions (at finite temperature). If we consider quantum phase transitions, we need to include the "imaginary time dimension" and $d_{RR} \to d_{RR} + 1$. In QPTs, the vanishing temperature makes fluctuations play a more critical role. Thus the effects of rare regions for quantum phase transitions are increased and become qualitatively different, the McCoy-Wu model, for example, exhibits smeared phases. I will not treat quantum phase transitions further.

Finally, one must mention that the dynamics inside the Griffiths phase are completely dominated by the rare regions for all classes. This is very important because Griffiths effects in short-range, uncorelated (point like) disorder are very weak. A way to confirm the formation of the Griffiths phase experimentally is by looking at the long-time dynamics of the classical phase transition system, the spin correlation function has a non-exponential decay [8].

3 The Dilute Ising Model

3.1 A nod to percolation theory

One of the first questions posed regarding percolation theory is the following: Submerge a porous rock into a tub of water, does its center get wet? The answer of course depends on how porous the rock is. In fact, one can be very quantitative by defining some quantities of interest. More importantly, the brief discussion here will provide a readily applicable description for the dilute Ising model.

In a nutshell, percolation has to do with the formation of clusters in a lattice. Imagine a d dimensional lattice, its sites can be either occupied or vacant with a probability p or $(1-p)$ respectively. A cluster is a group of nearest neighbor occupied sites. An occupied site that has only vacant sites as its neighbors is a cluster of size 1.

For a given hypercubic lattice of size L^d, we define a critical probability p_c. This is the minimum probability required for the formation of an infinite cluster (or in a finite lattice, for the cluster to touch two opposite boundaries of the lattice). Figure 2 shows percolation through a two dimensional lattice ($L \times L = 150 \times 150$) for different occupation probabilities. Grey sites are occupied, white sites are unoccupied, and the black sites are the largest cluster.

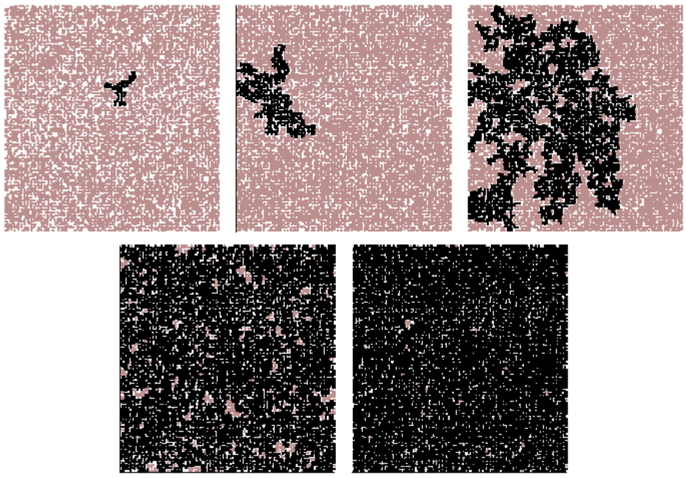

Figure 2: Percolation of a $2d$ square lattice with $L = 150$. Probabilities of occupation are $p = 0.45, 0.55, 0.59, 0.65, 0.75$. In this example, the largest cluster percolates from top to bottom for $p >= 0.59$. p_c for site percolation in the $2d$ infinite lattice converges to 0.5927. Taken from [9].

Different lattices in different dimensions (triangular, square, diamond, hypercube) converge to different critical ocupation probabilities. In general for a hypercubic latice in d dimensions, p_c decreases as the number of dimentions increase (it is harder for the cluster to form loops). Of course p_c is well defined only in the limit of the infinite lattice. A quick way to grasp this is thinking about an infinite one dimensional lattice chain. The probability of occupation must be unity in order for an infinite cluster to form, thus $p_c = 1$. In in a finite lattice, as illustrated in Figure 3 this is not the case (in a site of size 1 $p_c = p$). Here it is useful to define the strength of a cluster $\Pi(p, L)$, which is the probability for a random site belonging to the percolating cluster.

In the infinite limit, the percolation strength is written as $P(p)$ in the literature. It is evident that $P(p < p_c)$ vanishes. $P(p)$ is then the order parameter. Whenever $p \geq p_c$, $P(p)$ becomes non zero and our system experiences a phase transition.

I stop talking about percolation theory here, though there is much more to say about it. It will suffice to state for now that critical exponents can be defined, that they are different depending on the dimension of the lattice, and that they differ from

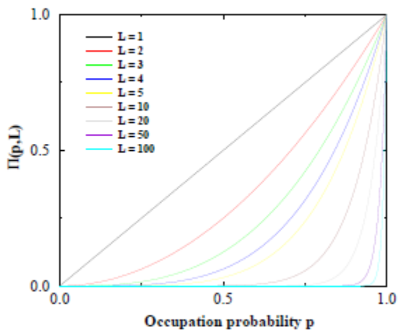

Figure 3: Probability of percolation for a $1d$ lattice of size L. In the infinite size limit, the probability of percolation converges to a step function (from [9]).

those of the Ising model in the same dimension. I urge the reader to dig a little and find out more about percolation theory. It is a very intuitive way to develop ideas about order, phase transitions, criticality, fractals (self-similarity of clusters occurs at $p = p_c$), and renormalization. Nice notes about it can be found in reference [9], and a good introductory book on percolation theory that can be found at the library [10]. An article worth looking at is [11] , it treats Ising model spins placed at the corners of fractal lattices, it is co-written by Mandelbrot.

3.2 The randomly diluted Ising model

In this section we treat the case of a latice where a fraction p of the sites is *vacant*, the rest being occupied by nearest neighbor interacting Ising spins. Clusters are groups of neighbouring occupied magnetic sites. Note that we changed the definition of p from occupied to vacant. Experimentally, we want p to be a knob that introduces disorder. In the lab, this knob could be pressure, or a the concentration of a dopant. The diluted Ising Hamiltonian is (for external field $h = 0$):

$$H = -J \sum_{\langle ij \rangle} \epsilon_i \epsilon_j S_i, S_j$$

Where $\langle ij \rangle$ sums over nearest neighbors only, $S_{i,j} = \pm 1$, $J > 0$ is the (ferromagnetic) interaction strength, and $\epsilon_{i,j}$ are take care of the dilution ($\epsilon_i = 1$ occupied, $\epsilon_i = 0$ vacant). The ϵ have a probability density (for point defects):

$$P(\epsilon) = (1 - p)\delta(\epsilon - 1) + p\delta(\epsilon)$$

Tha phase diagram is shown in [figure]. At $p = 0$ we find Ising exponents for our critical behavior. In the limit $T \to 0$ there is order for $p \leq p_c$. In this region we will find percolation exponents (and Ising exponents) for the magnetization m and the suceptibility χ. The percolation critical exponents can be found in [9],[10]. This hints at the effects of a Class B rare region, this makes sense because in the limit of vanishing temperature (QPT) fluctuatins play a more important role. The spins on a percolation cluster are parallel to each other.

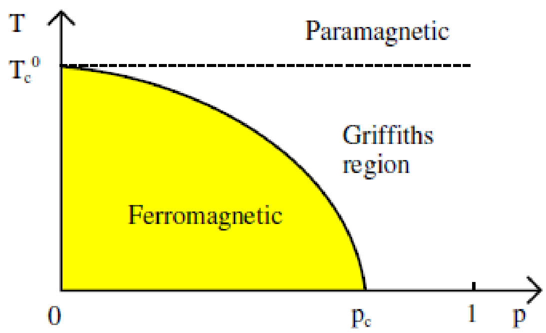

Figure 4: Phase diagram for a dilute ising ferromagnet. The horizontal axis is the probability that a site is *vacant*. The Griffiths phase is shown below the dashed line (Taken from [3])

4 Experimental Confirmations of the Griffiths phase

As discussed above, verifying the weak effects on the thermodynamic varibles caused by the Griffiths phase classical phase transitions is quite difficult [CITE 52]. Neverthless, some observations have been made in [CITE] on the phase of the antiferromagnet $FeBr_2$. Applying an external magnetic field H_a, reveasl fluctuating domain-like antiferromafnetic correlations avobe $T_c(H_a)$.

Using SQUID measurements, the experimentes detected a Griffiths phase in the region $T_c(H_a) < T < T_c(H_a = 0) = T_N^0$ by measuring the low frequency suceptibility χ Figure 5.

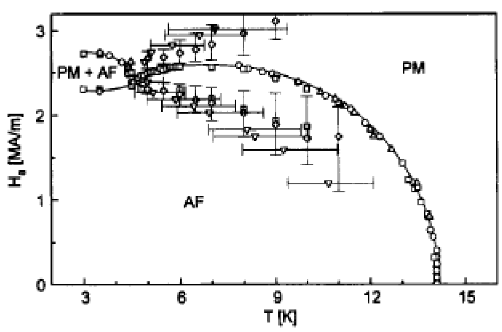

Figure 5: Phase diagram for $FeBr_2$t. Paramagnetic and antiferromagnetic phases are labeled.. The Griffiths phase is shown to dominate around $H_a = 2.67$ MA/m (Taken from [12])

In the Griffiths region fluctuations with non-critical bahavior for χ are seen as in Figure 6

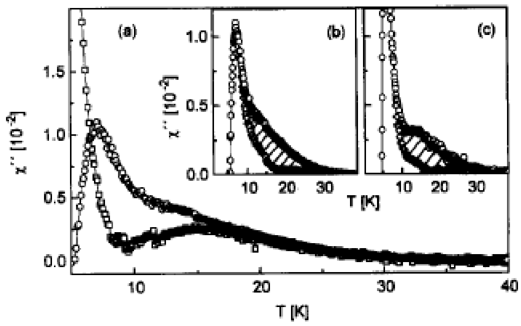

Figure 6: Phase diagram for $FeBr_2$t. Paramagnetic and antiferromagnetic phases are labeled.. The Griffiths phase is shown to dominate around $H_a = 2.67$ MA/m (Taken from [12])

5 Conclusions

I will discuss the the Griffiths phases and their effect on thermodynamic variables and critical phenomena were discussed in the context of rare regions. The presence of quenched disorder gives rise to significantly different behavior in magnetic spin systems. We classified the type of dissorder acording to its dimensionality and discussed the effects.

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