Quantum Hall States of Rapidly Rotating Bose-Einstein Condensates

Soheil Baharian

Loomis Laboratory of Physics
University of Illinois at Urbana-Champaign

May 5, 2006

Abstract

This essay is a review of the topic of vortices in rapidly rotating BECs, where the system is in a quantum Hall-like regime and the density profile is of the Thomas-Fermi form. Also, in this paper, deviations from a triangular lattice is being considered while the system condensates into the lowest Landau levels.

Contents

1 Introduction

In 2001, an experiment at MIT [1] resulted in creation of a large number ($\sim$ 100) of vortices in a Bose-Einstein condensate of alkali atoms rotating with frequency $\sim 0.36\omega_\perp$ where $\omega_\perp$ is the trap frequency. Since then, experiments have been performed [2] with rotation frequencies much higher than that of the MIT group's experiment, i.e. $\sim 0.99\omega_\perp$. In this regime, the radius of the trapped cloud increases, and so does the number of vortices; therefore, there exist many-vortex interactions instead of pair-wise ones which lead to emergence of exotic properties among which are the similarity to the quantum Hall liquids and the existence of a vortex lattice.
In this paper, we investigate, by quoting analytic [3, 4, 6] and numerical calculations [5, 6], the properties of the emerging vortex lattice in the fast rotation regime and determine the cloud's coarse-grained average density under the assumption that, although working in the fast rotation regime, the total number of vortices N_v is small compared to the total number of atoms N so that the ground state of the system could be described by a marcoscopic condensate wave function $\Psi(\mathbf{r})$ in what is known as the mean-field quantum Hall regime.

2 Physics Of A Particle In A Rotating Frame

Let the system rotate around the z axis with angular frequency Ω while being subject to a 2D harmonic trap in the $x-y$ plane. Then, the single-particle Hamiltonian in the rotating frame would be

$$\mathcal{H}_\perp = \frac{p_\perp^2}{2m} + \frac{1}{2}m\omega_\perp^2 r^2 - \Omega L_z \tag{1}$$

where $\mathbf{r} = (x, y)$, $\omega_\perp$ is the transverse trap frequency, $\mathbf{p}_\perp = -i\hbar\nabla_\perp$ is the momentum in the $x-y$ plane, and $L_z = -i\hbar\hat{\mathbf{z}} \cdot \mathbf{r} \times \nabla_\perp$ is the angular momentum around the z axis. Adding and subtracting $\omega_\perp L_z$, we could rewrite the Hamiltonian as

$$\mathcal{H}_\perp = \frac{1}{2m}\left(-i\hbar\nabla_\perp - m\omega_\perp\hat{\mathbf{z}} \times \mathbf{r}\right)^2 + (\omega_\perp - \Omega)\, L_z. \tag{2}$$

2.1 The energy spectrum

The first term in (2) is similar to the Hamiltonian of a charged particle moving in the $x-y$ plane in a magnetic field, i.e. $\frac{1}{2m}\left(-i\hbar\nabla_\perp - e\mathbf{A}/c\right)^2$.

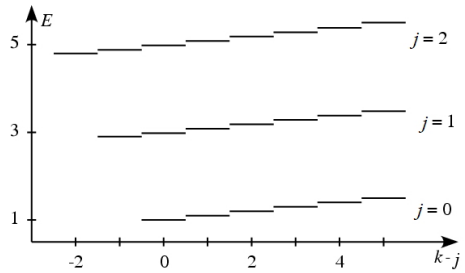


Figure 1: Landau levels grouped by the index j for $\Omega = 0.9\omega_\perp$; the energy is in units of $\hbar\omega_\perp$.

Thus, its normalized eigenfunctions are the Landau levels

$$\phi_{j,k}(\mathbf{r}) = \frac{e^{|u|^2/2d_\perp^2}\partial_+^k\partial_-^j e^{-|u|^2/d_\perp^2}}{\sqrt{\pi d_\perp^2 j!k!}} \tag{3}$$

with eigenvalues $\hbar\omega_\perp(2j+1)$ where $d_\perp = \sqrt{\hbar/m\omega_\perp}$ is the transverse oscillator length, $u = x + iy$, $\partial_\pm = (d_\perp/2)(\partial_x \pm i\partial_y)$, and j and k are integers characterizing the Landau level index and the degenerate states within a Landau level respectively. However, $\phi_{j,k}$ are also eigenstates of L_z with eigenvalues $\hbar(k-j)$; therefore, the eigenvalues of $\mathcal{H}_\perp$ are just

$$\hbar\big((\omega_\perp + \Omega)j + (\omega_\perp - \Omega)k + \omega\big). \tag{4}$$

Note that, as can be seen in figure 1, the degeneracy has been removed and that, when $\Omega \approx \omega_\perp$, there exists an energy gap of $\mathcal{O}(2\hbar\omega_\perp)$ between the two adjacent Landau levels while two states in a given Landau level are separated by a gap of $\mathcal{O}\big(\hbar(\omega_\perp - \Omega)\big) \ll \hbar\omega_\perp$.
It is obvious that the rotation frequency Ω must be smaller than the transverse trapping frequency $\omega_\perp$ in order to prevent the particle from escaping the trap.

2.2 The lowest Landau levels

In section 3.2, it will be shown the system will reside in the states characterized by $j = 0$, namely the *lowest Landau levels (LLL)* (see figure 1). Since $\phi_{0,k}(\mathbf{r}) \propto u^k e^{-r^2/2d_\perp^2}$, any general wave function in the LLL function space is a linear combination of $\phi_{0,k}$'s and can be written as $\phi_{\mathrm{LLL}}(\mathbf{r}) = Ce^{-r^2/2d_\perp^2}p(u)$ where $p(u)$ is a polynomial in (and therefore an analytic

function of) u and C normalizes $\phi_{\mathrm{LLL}}(\mathbf{r})$ to unity. Using the fundamental theorem of algebra then leads to

$$\phi_{\mathrm{LLL}}(\mathbf{r}) = Ce^{-r^2/2d_\perp^2} \prod_\alpha (u - u_\alpha) \tag{5}$$

where u_α are complex zeros of $p(u)$. It's obvious from (5) that going around each u_α changes the phase of $\phi_{\mathrm{LLL}}(\mathbf{r})$ by by 2π, so each u_α is the position of a single-charge vortex.
A way to interpret (5) is to write $|\phi_{\mathrm{LLL}}(\mathbf{r})|^2 = |C|^2 e^{-U(\mathbf{r})}$ where

$$U(\mathbf{r}) = \frac{r^2}{d_\perp^2} - 2\sum_\alpha \ln|\mathbf{r} - \mathbf{r}_\alpha| \tag{6}$$

could be interpreted as the energy of a unit-charge particle interacting in two dimensions with a positive uniform background charge of density $\rho = 1/\pi d_\perp^2$ and a set of $q_\alpha = -1$ charges located at $\mathbf{r}_\alpha$, written in Gaussian units [3]. This can easily be seen by rewriting (6) as

$$\nabla_\perp^2 U(\mathbf{r}) = 4\pi \left(\frac{1}{\pi d_\perp^2} - \sum_\alpha \delta(\mathbf{r} - \mathbf{r}_\alpha) \right) \tag{7}$$

which is just Gauss's law in two dimensions for the system described above [3].
Another way to look at (6) is to define the atomic density $n_a(\mathbf{r}) = \left|\frac{\phi_{\mathrm{LLL}}(\mathbf{r})}{C}\right|^2$ and the vortex density $n_v(\mathbf{r}) = \sum_\alpha \delta(\mathbf{r} - \mathbf{r}_\alpha)$ and note that

$$\nabla_\perp^2 \ln n_a(\mathbf{r}) = 4\pi \left(-\frac{1}{\pi d_\perp^2} + n_v(\mathbf{r}) \right) \tag{8}$$

which shows that there's a one-to-one relation between the atomic and vortex densities in the LLL regime [4, 6].

3 Interactions In A Rotating Gas

Let's assume that there exist interactions between the atoms of a cold gas of N bosons with mass m which is subject to a cylindrically symmetric harmonic potential with frequencies $\omega_\perp$ in the $x-y$ plane and ω_z along the z axis.
If $\hbar\omega_z$ is very large compared to all other energy scales in the system, we can safely assume that the system resides in the ground state of the harmonic

trap in the z direction where $d_z = \sqrt{\hbar/m\omega_z}$ is the axial extent of the cloud in that direction. Thus, the gas is now a quasi-two-dimensional system rotating around the z axis with frequency Ω close to $\omega_\perp$.
The macroscopic wave function of the system $\Psi(\mathbf{r})$ is determined by the Gross-Pitaevskii energy functional. If we define $\Psi(\mathbf{r}) = \sqrt{N}\psi(\mathbf{r})$ where $\psi(\mathbf{r})$ is normalized to unity, then the energy per particle in the rotating frame would be

$$E[\psi] = \int \left(\psi^* \mathcal{H}_\perp \psi + \frac{1}{2} g_{2\mathrm{D}} |\psi|^4 \right) \mathrm{d}^2 r \tag{9}$$

where $\mathcal{H}_\perp$ is defined in (2) and $g_{2\mathrm{D}} = Ng/\sqrt{2\pi} d_z$ is the effective coupling strength in two dimensions where $g = 4\pi\hbar^2 a_s/m$ is the two-body interaction strength and a_s is the s-wave scattering length. Minimizing (9) with respect to $\psi(\mathbf{r})$ would lead to the Gross-Pitaevskii equation

$$\left(\mathcal{H}_\perp + g_{2\mathrm{D}} |\psi(\mathbf{r})|^2 \right) \psi(\mathbf{r}) = \mu \psi(\mathbf{r}) \tag{10}$$

where the chemical potential μ ensures that $\psi(\mathbf{r})$ is normalized to unity, i.e. it is a Lagrange multiplier for the constraint $\int |\psi(\mathbf{r})|^2 \mathrm{d}^2 r = 1$.

3.1 The LLL regime in the presence of interactions

Because of the presence of interactions, the states $\phi_{j,k}(\mathbf{r})$ are no longer the eigenstates of the system. This means that interactions mix different (j,k) Landau levels. However, as will be seen later in section 3.2, for a given $g_{2\mathrm{D}}$ and for sufficiently fast rotations, i.e. $\Omega \approx \omega_\perp$, the system resides in the lowest Landau levels with good accuracy. In fact, as Ω approaches $\omega_\perp$ (as the number of vortices increase), the trapping force $-m\omega_\perp^2 r$ is nearly balanced by the centrifugal force $m\Omega^2 r$ and, therefore, the cloud expands and the distance between the particles increases. Hence, the effect of interactions gets smaller and the system looks more and more like the non-interacting case where the energy per particle is $\hbar\omega_\perp$.
Choosing $\psi(\mathbf{r})$ to be in the LLL, the angular momentum per particle could be written

$$\langle L_z \rangle = \hbar \int \left(\frac{r^2}{d_\perp^2} - 1 \right) |\psi(\mathbf{r})|^2 \mathrm{d}^2 r \tag{11}$$

plus terms of $\mathcal{O}\left(\frac{1}{N_v}\right)$ where N_v is the total number of vortices [4]. Thus, in the LLL regime, (9) could be written as [3, 4, 6]

$$E[\psi] = \hbar\Omega + \int \left(\hbar(\omega_\perp - \Omega) \frac{r^2}{d_\perp^2} |\psi(\mathbf{r})|^2 + \frac{1}{2} g_{2\mathrm{D}} |\psi(\mathbf{r})|^4 \right) \mathrm{d}^2 r \tag{12}$$

or, when rescaled [6], as

$$\epsilon\left[\psi\right]=\frac{E\left[\psi\right]-\hbar\Omega}{\hbar\left(\omega_{\perp}-\Omega\right)}=\int\left(\frac{r^{2}}{d_{\perp}^{2}}\left|\psi\left(\mathbf{r}\right)\right|^{2}+\frac{1}{2}\Lambda\left|\psi\left(\mathbf{r}\right)\right|^{4}\right)\mathrm{d}^{2}r \tag{13}$$

where

$$\Lambda=\frac{g_{2\mathrm{D}}}{\hbar\left(\omega_{\perp}-\Omega\right)}. \tag{14}$$

As can be seen in (13), when the reduced energy $\epsilon\left[\psi\right]$ is minimized in the LLL regime, $\psi\left(\mathbf{r}\right)$ will depend only on one parameter, Λ, rather than on two separate parameters for the general case in (9), namely $g_{2\mathrm{D}}$ and Ω.

3.2 Validity of the LLL approximation

Minimizing $\epsilon\left[\psi\right]$ with respect to $\psi\left(\mathbf{r}\right)$, i.e. $\frac{\delta\epsilon\left[\psi\right]}{\delta\psi}=0$, with the *only* constraint that $\psi\left(\mathbf{r}\right)$ be normalized to unity, we will find after some algebra

$$\left|\psi_{\min}\left(\mathbf{r}\right)\right|^{2}=\frac{2}{\pi R^{2}}\left(1-\frac{r^{2}}{R^{2}}\right) \tag{15}$$

where $R=\left(\frac{2\Lambda d_{\perp}^{2}}{\pi}\right)^{1/4}$ is the radius of the cloud, i.e. $\left|\psi\left(\mathbf{r}\right)\right|^{2}=0$ for $r>R$ [6]. The reduced energy, then, would be

$$\epsilon_{\min}=\sqrt{\frac{8}{9\pi}}\sqrt{\frac{\Lambda}{d_{\perp}^{2}}}. \tag{16}$$

It is clear from (15) that $\psi_{\min}\left(\mathbf{r}\right)$ does not belong to the LLL - the only function in the LLL function space which depends only on r is $\exp\left(-r^{2}/2d_{\perp}^{2}\right)$ - since we had the freedom of choosing the most suitable function to minimize the reduced energy functional without any restrictions on the funtion space. Therefore, the reduced energy $\epsilon_{\min}$ is lower than $\epsilon\left[\psi\right]$ when $\psi\left(\mathbf{r}\right)$ varies only in the LLL function space. So, while minimizing $\epsilon\left[\psi\right]$ in the LLL function space, we should look for a function which is *the closest* to $\psi_{\min}\left(\mathbf{r}\right)$ so that its corresponding reduced energy is *the closest* to $\epsilon_{\min}$.
As will be seen in chapter 4, the result of minimizing the reduced energy functional in all cases would be $\epsilon\left[\psi\right]=c\sqrt{\Lambda/d_{\perp}^{2}}$ where c is $\mathcal{O}\left(1\right)$. Thus, according to (13), the energy of the ground state of the rotating cloud $E\left[\psi\right]$ would be $\hbar\Omega+c\hbar\left(\omega_{\perp}-\Omega\right)\sqrt{\Lambda/d_{\perp}^{2}}$. So if the excess energy $c\hbar\left(\omega_{\perp}-\Omega\right)\sqrt{\Lambda/d_{\perp}^{2}}$ is small compared to the separation between the LLL

and the first excited Landau level, i.e. $2\hbar\omega_\perp$, then the assumption that the system resides in the LLL is valid. After some algebra, we find

$$\frac{mg_{2\mathrm{D}}}{\hbar^2}\left(1-\frac{\Omega}{\omega_\perp}\right)\ll 1 \tag{17}$$

which is quite easy to satisfy since $\Omega\to\omega_\perp$.

4 The Vortex Lattice

Our goal, in this section, is to minimize the reduced energy functional in the LLL regime with different lattice configurations. First, we consider the case of an infinite regular lattice and its effect on the atomic density profile. Then, we let the vortex lattice relax from its uniform configurations and calculate the density profile associated with this distorted lattice.

4.1 Regular lattice : averaged vortex approximation

Looking back at (7), since $\{\mathbf{r}_\alpha\}$ designates an infinite regular lattice, we could write [3] $\sum_\alpha \delta\left(\mathbf{r}-\mathbf{r}_\alpha\right)=\mathcal{A}^{-1}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}$ where $\mathcal{A}$, the size of the unit cell, is equal to $\frac{1}{\overline{n_v}}$ where $\overline{n_v}$ is the average vortex density. So, since $U\left(\mathbf{r}\right)$ should be real, we could write

$$\nabla_\perp^2 U\left(\mathbf{r}\right)=4\left(\frac{1}{d_\perp^2}-\frac{\pi}{\mathcal{A}}\right)-\frac{4\pi}{\mathcal{A}}\sum_{\mathbf{k}\neq 0}\cos\mathbf{k}\cdot\mathbf{r}. \tag{18}$$

Now, defining

$$\frac{1}{\sigma^2}=\frac{1}{d_\perp^2}-\frac{\pi}{\mathcal{A}}, \tag{19}$$

$$\overline{n_a}\left(r\right)=\frac{1}{\pi\sigma^2}e^{-r^2/\sigma^2}, \tag{20}$$

and

$$f\left(\mathbf{r}\right)=\prod_{\mathbf{k}\neq 0} e^{-\frac{4\pi}{\mathcal{A}k^2}\cos\mathbf{k}\cdot\mathbf{r}}, \tag{21}$$

we could write [3, 6]

$$n_a\left(\mathbf{r}\right)=\overline{n_a}\left(r\right)f\left(\mathbf{r}\right) \tag{22}$$

where $n_a\left(\mathbf{r}\right)$ is the atomic density defined in section 2.2, $\overline{n_a}\left(r\right)$ is the coarse-grained average of the atomic density over a linear size large compared with

the vortex separation but small compared with σ, and $f(\mathbf{r})$ is a periodic function over the lattice which vanishes at $\mathbf{r}_\alpha$'s [1].
As (21) suggests, $f(\mathbf{r})$ contains information mainly about the lattice type and, therefore, its contribution in determining the overall density profile is small. Thus, we could ignore $f(\mathbf{r})$ in (22) and replace $n_a(\mathbf{r})$ by its coarse-grained average $\overline{n_a}(r)$, namely the *averaged vortex approximation* [3].
Now that the overall behavior of the density profile is known using the the averaged vortex approximation, the reduced energy could be written as [4, 6]

$$\epsilon_{\text{ava}} \approx \int \left(\frac{r^2}{d_\perp^2} \overline{n_a}(r) + \frac{1}{2} b \Lambda \overline{n_a}^2(r) \right) \mathrm{d}^2 r \tag{23}$$

where $b = \overline{n_a^2}(r) / \overline{n_a}^2(r) \approx 1.159$ is the Abrikosov parameter for the triangular lattice. We should note that the interaction coefficient is *renormalized* by a factor of b since instead of working with the exact atomic density $n_a(\mathbf{r})$, we're only dealing with its coarse-grained average $\overline{n_a}(r)$.
Minimizing ϵ_{ava} with respect to σ, we find that

$$\epsilon_{\text{ava}} = \sqrt{\frac{b}{\pi}} \sqrt{\frac{\Lambda}{d_\perp^2}}. \tag{24}$$

This reduced energy is greater than the one in (16) by a factor of $\sqrt{9b/8} \approx 1.14$ which is due to (a) coarse-grained averaging of the atomic density and (b) using an atomic density with a Gaussian profile rather than the optimum profile of an inverted parabola.

4.2 Distorted lattice : Thomas-Fermi profile

Going back to (13), we could use the averaged vortex approximation here to obtain an approximate form for the reduced energy [4]

$$\epsilon_{\text{TF}} \approx \int \left(\frac{r^2}{d_\perp^2} \left\langle |\psi(\mathbf{r})|^2 \right\rangle + \frac{1}{2} b \Lambda \left\langle |\psi(\mathbf{r})|^2 \right\rangle^2 \right) \mathrm{d}^2 r \tag{25}$$

where $\langle \cdots \rangle$ denotes averaging over a linear size large compared with the vortex separation but small compared with the radius of the cloud. The interaction coefficient is renormalized again to take into account the effects

[1] Let us define the regular lattice as $\mathbf{r}_{\alpha_1,\alpha_2} = \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2$ where $\mathbf{a}_1$ and $\mathbf{a}_2$ are the lattice primitive vectors (α_1 and α_2 are integers). Then the reciprocal lattice primitive vectors are $\mathbf{k}_1 = (2\pi/\mathcal{A})\, \mathbf{a}_2 \times \hat{\mathbf{z}}$ and $\mathbf{k}_2 = (2\pi/\mathcal{A})\, \hat{\mathbf{z}} \times \mathbf{a}_1$ where $\mathcal{A} = |\mathbf{a}_1 \times \mathbf{a}_2|$ is the size of the unit cell. Thus, $\mathbf{k} = \beta_1 \mathbf{k}_1 + \beta_2 \mathbf{k}_2$ (β_1 and β_2 are integers) [3].

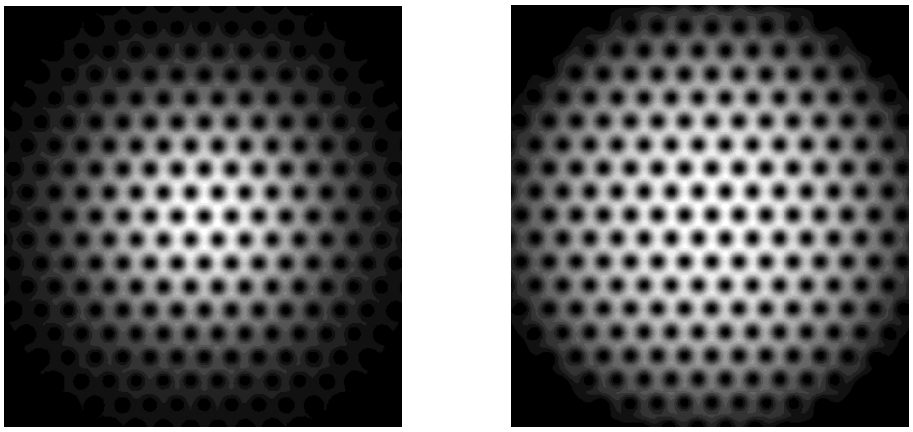
of this coarse-grained averaging of the density profile.
One might be tempted to think that ϵ_{TF} obtained above is just the one we have already had in (23), but there exists one important difference: in (23), we had obtained a Gaussian form for the smoothed density profile, namely $\overline{n_a}(r)$, by imposing the constraint of the lattice being triangular and infinite before minimizing the reduced energy, while here, we do not impose any constraint on the lattice and, therefore, are just minimizing ϵ_{TF} with respect to $\left\langle |\psi(\mathbf{r})|^2 \right\rangle$.
After some algebra, the coarse-grained average of the atomic density that minimizes (25) is found to be [4, 5, 6]

$$\left\langle |\psi_{\mathrm{TF}}(\mathbf{r})|^2 \right\rangle = \frac{2}{\pi R_{\mathrm{TF}}^2}\left(1 - \frac{r^2}{R_{\mathrm{TF}}^2}\right) \tag{26}$$

where $R_{\mathrm{TF}} = \left(\frac{2b\Lambda d_\perp^2}{\pi}\right)^{1/4}$ is the radius of the cloud. Note that (26) looks exactly like (15); however, unlike (15) which is not a LLL wave function, there exists a $\psi_{\mathrm{TF}}(\mathbf{r})$ in the LLL function space whose *coarse-grained average* is an inverted parabola. Using (26) to calculate ϵ_{TF} leads to

$$\epsilon_{\mathrm{TF}} = \sqrt{\frac{8b}{9\pi}}\sqrt{\frac{\Lambda}{d_\perp^2}}. \tag{27}$$

It's worth noting that (27) is larger than the reduced energy in (16) by just a factor of $\sqrt{b} \approx 1.07$.
Since $\epsilon_{\mathrm{min}} < \epsilon_{\mathrm{TF}} < \epsilon_{\mathrm{ava}}$, it is energetically favorable for the system to distort the lattice rather than be in a regular one. In other words, changing the amplitudes of the LLL components of the wave function without exciting any higher Landau levels leads to changing the positions of the vortices from a regular lattice. Thus, the smoothed density profile $\left\langle |\psi_{\mathrm{TF}}(\mathbf{r})|^2 \right\rangle$ and the reduced energy ϵ_{TF} are *the closest* that we could get to the optimum solutions while remaining in the LLL function space.
Numerical calculations [5, 6] also confirm these analytical results (see figure 2). It is easy to notice that figure 2a has a Gaussian profile which dies off more rapidly than the profile in figure 2b which is an inverted parabola. Also, figure 3 shows a more quantitative comparison of the two lattice types where the coarse-grained average atomic density obtained by minimizing the reduced energy while letting the lattice relax from a triangular one is closely matched by the Thomas-Fermi profile. Also, the smoothed density profile of a regular lattice is reprodued by the Gaussian profile.
The amount of deformation of the lattice from a regular one can be estimated



(a) (b)

Figure 2: The atomic density for (a) triangular lattice and (b) distorted lattice [5].

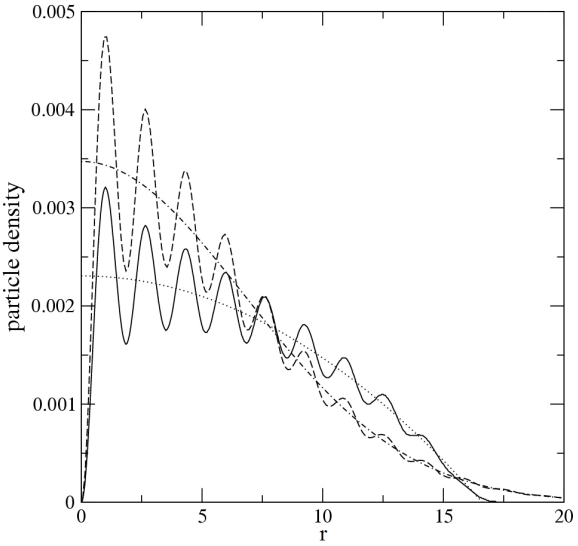


Figure 3: Atomic density (in units of $d_\perp^2$) as a function of r (in units of $d_\perp$). The solid line corresponds to the atomic density in a distorted lattice and the dashed line to a triangular lattice. The dotted line is the Thomas-Fermi profile and the dot-dash line is the Gaussian profile [5].

using (8) for a general density distribution. Thus, assuming a Thomas-Fermi form for the smoothed atomic density, we find [4]

$$n_v(r) = \frac{1}{\pi d_\perp^2} - \frac{1}{\pi R_{\rm TF}^2 \left(1 - r^2/R_{\rm TF}^2\right)^2}. \tag{28}$$

The second term above is $\mathcal{O}\left(\frac{1}{N_v}\right)$ compared with the first, since $N_v \approx R_{\rm TF}^2/d_\perp^2$. This means that very small deviations from a regular lattice, i.e. small changes in the vortex density, can cause very large variations in the smoothed atomic density profile in the LLL regime.

5 Conclusion

We have done analytical and shown numerical calculations of physical quantities of vortex lattices in rapidly rotating cold weakly-interacting atomic Bose-Einstein condensates. We have shown that at the rotation rate increases, the gas condensates into the lowest Landau levels and becomes quantum Hall-like. The results presented in this paper show that the vortex lattice distorts slightly from a triangular configuration and that the atomic density averaged over a length scale larger than the vortex separation is given by a Thomas-Fermi profile. All these results are experimentally verifiable and have indeed been confirmed. It should also be noted that the results of this paper are true for regions away from the boundaries of the cloud.

References

[1] J. R. Abo-Shaeer, C. Raman, J. M. Vogels, W. Ketterle, Science **292**, 476-479 (2001)

[2] V. Schweikhard, I. Coddington, P. Engels, V. P. Mogendroff, E. A. Cornell, Phys. Rev. Lett. **92**, 040404 (2004)

[3] Tin-Lun Ho, Phys. Rev. Lett. **87**, 060403 (2001)

[4] Gentaro Watanabe, Gordon Baym, C. J. Pethick, Phys. Rev. Lett **93**, 190401 (2004)

[5] N. R. Cooper, S. Komineas, N. Read, Phys. Rev. A **70**, 033604 (2004)

[6] Amandine Aftalion, Xavier Blanc, Jean Dalibard, Phys. Rev. A **71**, 023611 (2005)