Superfluid ^3He

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Abstract

In this report I will discuss the main properties of the superfluid phases of Helium 3. First, a brief description of the experimental observations and the phase diagram of Helium 3 is made. This is followed by a discussion of the superfluid phases from a theoretical point of view based on the generalized BCS theory in the weak coupling regime.

1 Introduction

Helium, the second element in the periodic table, is composed of a 2 proton nucleus surrounded by a closed electronic shell. Two stable isotopes are known: ^3He and ^4He, the former is a fermion with nuclear spin of 1/2 while the latter is a boson of zero spin. This difference in nuclear spin has deep consequences in their low temperature behavior. While both isotopes undergo a transition to a superfluid state at sufficiently low temperatures, the mechanisms responsible for the transition are completely different. On the one hand, ^4He atoms are bosons so they achieve superfluidity by Bose-Einstein condensation and on the other hand, ^3He atoms are fermions so they undergo a BCS-type transition and achieve superfluidity by the formation of paired electronic states called Cooper pairs.

Helium atoms interact via Van der Waals-type forces, because of their closed electronic shells, with a hard core repulsion at short distance and an atractive interaction at long range. An effective attractive interaction is needed to form Cooper pairs at low temperature, so this rules out the possibility of having s-wave pairing in the system because of the large overlap of the wavefunctions of such a state at zero separation, where the $He - He$ interaction is highly repulsive. In order to use the attractive part of the interaction to produce pairing, the Copper pairs must be in a state of nonzero relative angular momentum. As will be discussed below in greater detail, the Cooper pairs in ^3He are in a spin-triplet state with a relative angular momentum $l = 1$.

The paper is divided as follows. First a brief description of some of the most important results from experimental observations, including the low temperature phase diagram, the specific heat and the magnetic susceptibility is presented. This is followed by a short introduction to the weak coupling generalized BCS theory. Finally, I discuss the main results of the theory corresponding to the zero magnetic field phases.

2 Experimental Properties

Figure 1 shows the low temperature phase diagram of ^3He in the P-T plane at zero external magnetic field [1, 2]. Aside from the normal liquid state, which is successfully described by Landau's Fermi Liquid Theory, 2 additional liquid phases have been identified; both of them are superfluid. Below a critical pressure, commonly known as the polycritical point (P_{PCP}), there is a second order phase transition from the normal liquid state to a state commonly known as the B-phase. This state is stable all the way to zero temperature for $P < P_{PCP}$.

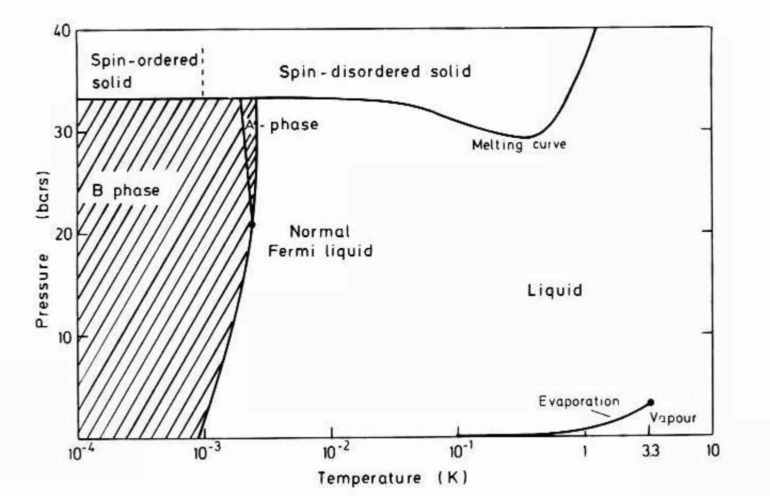

Figure 1: Low temperature phase diagram of ^3He in the P-T plane

Above the critical pressure, a new phase emerges between the normal state and the B-phase. This new phase, known as the A-phase, is also superfluid and the transition is second order. As the temperature is decreased further, there is a first order transition to the B-phase.

The application of an external magnetic field significantly alters the structure of the phase diagram. Figure 2 shows the P-T-H phase diagram. For non-zero magnetic field, the A-phase is stabilized all the way to zero pressure and the transition between the normal phase and the A-phase is splitted into two transitions, with the emergence of a new phase known as the A_1-phase.

The thermal properties of the new phases are drastically different from those of the normal state. Figure 3 shows the specific heat at zero magnetic field and at melting pressure as a function of T/T_c, where T_c is the critical temperature in the normal-A transition. As can be seen, there is a sharp discontinuity in the specific heat at T_c, consistent with the 2^{nd} order phase transition. There are no significant features in the A-B transition, on the other hand. The relative size of the discontinuity, $\frac{\Delta C_V}{C_N}$, can be as large as 2 and depends on pressure.

The magnetic properties of the new phases are also very different from the normnal state. Figure 4 shows the spin susceptibility in the B-phase as a function of T/T_c for a pressure below P_{PCP}. As can be seen, the spin susceptibility is temperature dependent. It follows a universal behaviour independent of pressure. This is consistent with the idea of p-wave pairing in a spin-triplet state as proposed by Balian and Werthamer [6], as we'll see below.

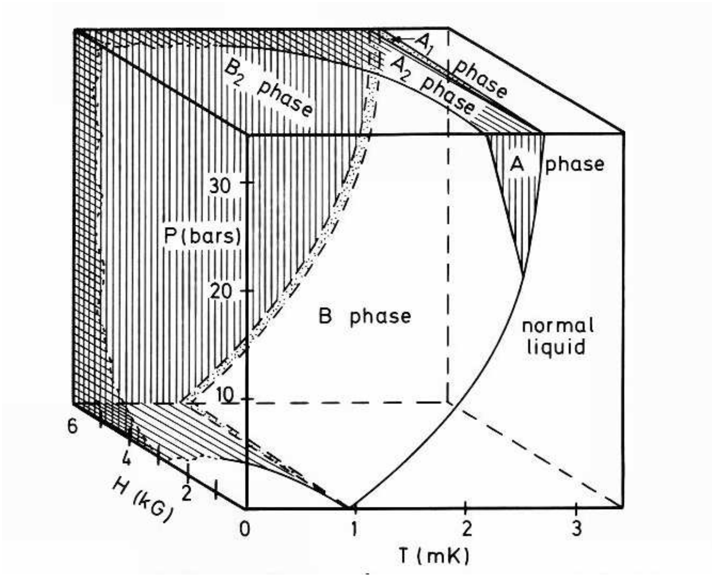

Figure 2: P-T-H phase diagram of ^3He.

3 Theoretical Description

Soon after the creation of the BCS theory of superconductivity in 1957 [4], several authors suggested that a similar transition should occur in ^3He and that it too should become superfluid at low enough temperatures. It was soon realized that s-wave pairing was not an option in this system because of the strong hard core repulsion between He atoms, so higher order pairing must be considered.

After many years of careful study of this system, the theoretical consensus is that in the superfluid phases Cooper pairs form in spin-triplet states with relative angular momentum l=1 [3]. As a consequence of this, the Cooper pairs in ^3He have an internal structure, which is directly responsible for the multiple phases at low temperature. To understand how this internal structure of the pairs is responsible for the different phases, let's consider the wave-function for a pair of atoms in a spin-triplet state and relative angular momentum l=1 [1]:

$$|\Psi >= \Psi_{1,+}(k)| \uparrow\uparrow> +\Psi_{1,0}(k)\,(| \uparrow\downarrow> +| \downarrow\uparrow>) + \Psi_{1,-}(k)| \downarrow\downarrow> \ . \quad (1)$$

In the B-phase, generally associated with the BW state [6], the three spin states $\Psi_{1,+}(k)$, $\Psi_{1,0}(k)$ and $\Psi_{1,-}(k)$ are populated with

[1] In this section, k, k' will refer to Fourier components, while Greek indexes refer to spin components.

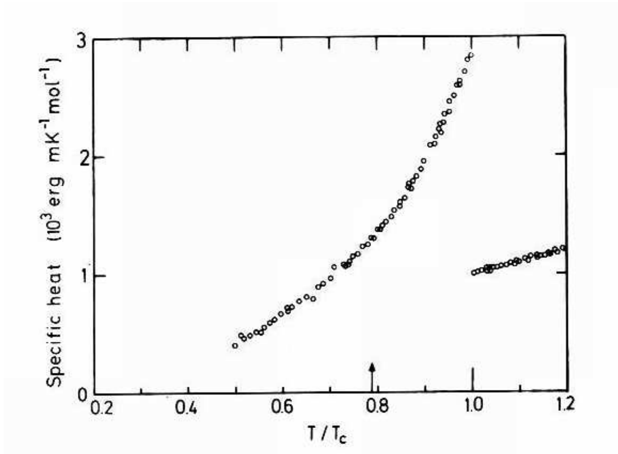

Figure 3: Specific heat at zero magnetic field and melting pressure as a function of T/T_c.

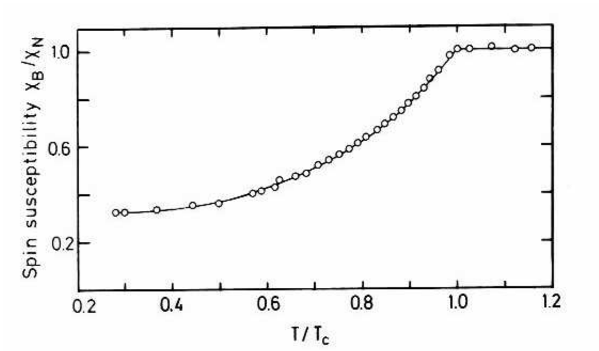

Figure 4: Magnetic Susceptibility of the B-phase as a function of T/T_c. The pressure is below the P_{PCP} critical point.

the same probability, so we have a homogeneous mixture. In the A-phase, asiociated with the ABM state [5, 7], the population of the state with S_z=0 is completely depleted, so all states have a non-zero spin projection along z. At zero magnetic field, the mixture of this states is homogeneous. With the application of a external magnetic field, the population of the state with S_z=-1 becomes zero and the system becomes spin polarized. This is the A_1-phase discussed before.

For low temperatures, the BCS theory can be generalized to non s-wave pairing in the weak coupling regime. In the rest of the paper, I briefly discuss the generalized theory and describe some of the main results appropriate to the BW and ABM states.

3.1 Weak-Coupling: Generalized BCS Theory

In order to study the superfluid phases of ^3He, we need to generalized the BCS theory to the case of non s-wave pairing. To do this, we start by defining the generalized BCS wavefunction:

$$|\Psi >= \prod_k \prod_\alpha \left(u_{k\alpha\alpha} + \sum_\beta v_{k\alpha\beta} a^+_{k\alpha} a^+_{-k\beta} \right) |0> \quad , \tag{2}$$

where $a^+_{k\alpha}$, $a_{k\alpha}$ are the usual creation and annihilation operators and $v_{k\alpha\beta}$, $u_{k\alpha\beta}$ are variational parameters to be determined by energy minimization. The Hamiltonian has the form:

$$H - \mu N = \sum_{k,\alpha} \epsilon_{k\alpha} a^+_{k\alpha} a_{k\alpha} + \tag{3}$$

$$\frac{1}{2} \sum_{k,k',q} \sum_{\alpha,\beta,\alpha',\beta'} \left\langle -k\alpha, k + q\beta |V| k^{'}\alpha^{'}, -k^{'} + q\beta^{'} \right\rangle a^+_{k'\alpha'} a^+_{-k'+q\beta'} a_{-k\alpha} a_{k+q\beta} \quad ,$$

where $\epsilon_{k\alpha} = \epsilon_k - \alpha\mu_o H = \frac{\hbar^2 k^2}{2m} - \mu - \alpha\mu_o H$ are the free particle energy states for a finite magnetic field H. Because we are assuming that the coupling is weak, we can simplify the Hamiltonian by making a mean field approximation, by replacing products of operators, a_k $a_{k'}$ or a^+_k $a^+_{k'}$, by their average value. If we introduce the off-diagonal mean field (or gap parameter as we'll see) defined as:

$$\Delta_{k\alpha\beta} = \sum_{k',\alpha',\beta'} \left\langle -k\alpha, k\beta |V| k^{'}\alpha^{'}, -k^{'}\beta^{'} \right\rangle \left\langle a_{-k'\alpha'} a_{k'\beta'} \right\rangle \quad , \tag{4}$$

the mean field Hamiltonian, H_{MF} becomes:

$$H_{MF} - \mu N = \sum_{k,\alpha} \epsilon_{k\alpha} a^+_{k\alpha} a_{k,\alpha} + \frac{1}{2} \sum_{k,\alpha,\beta} \left(\Delta^*_{k\alpha\beta} a_{-k\beta} a_{k\alpha} \right. \tag{5}$$

$$\left. + \, a^+_{k\alpha} a^+_{-k\alpha} \Delta_{k\alpha\beta} \right) - \frac{1}{2} \sum_{k,\alpha,\beta} \Delta^*_{k\alpha\beta} \left\langle a_{-k\beta} a_{k\alpha} \right\rangle \quad .$$

As can be seen, with this approximation the Hamiltonian becomes second order and can be diagonalized by making the usual Bogoliubov transformation:

$$b_{k\alpha} = \sum_\beta \left(u_{k\alpha\beta} a_{k\beta} - v_{k\alpha\beta} a^+_{-k\beta} \right) \tag{6}$$

$$b^+_{k\alpha} = \sum_\beta \left(u^*_{k\alpha\beta} a^+_{k\beta} - v^*_{k\alpha\beta} a_{k\beta} \right) \quad . \tag{7}$$

The $b_{k\alpha}, b_{k\alpha}^{+}$ are quasi-particle operators obeying standard anti-commutation relations. This is very similar to the original BCS theory, with the difference that the variational parameters u_k and v_k become spin dependent and mixed to allow for p-wave pairing. In terms of these operators, the digonalized mean field Hamiltonian becomes:

$$H_{MF} - \mu N = \sum_{k,\alpha} \left[-\frac{1}{2} \left(\epsilon_{k\alpha} + E_{k\alpha} \right)^{-1} \left(\Delta_k \Delta_k^{+} \right)_{\alpha\alpha} \right. \tag{8}$$

$$\left. + E_{k\alpha} b_{k\alpha}^{+} b_{k\alpha} - \frac{1}{2} \sum_{\beta} \Delta_{k\alpha\beta}^{*} v_{k\alpha\beta} u_{k\alpha\beta}^{*} \right] \quad ,$$

where $E_{k\alpha} = \sqrt{\epsilon_{k\alpha}^2 + (\Delta_k \Delta_k^{+})^2}$ are the quasiparticle energy levels. As expected, there is a gap in the quasi-particle excitation spectrum, but in this case the magnitude of the gap depends on the direction in $\mathbf{k}$ space, leading to possible anisotropic effects. This is typical of non s-wave pairing, and we will see a concrete example below when we consider the ABM state asociated with the A phase of ^3He.

The excitations of the above Hamiltonian are the *independent* quasi-particles described by the Bogoliubov operators. At finite temperature, the average number of these excitations is given by the standard Fermi distribution:

$$\left\langle b_{k\alpha}^{+} b_{k'\alpha'} \right\rangle = \delta_{kk'} \delta_{\alpha\alpha'} f_{k\alpha} \quad , \tag{9}$$

$$\left\langle b_{k\alpha}^{+} b_{k'\alpha'}^{+} \right\rangle = \left\langle b_{k\alpha} b_{k'\alpha'} \right\rangle = 0 \quad , \tag{10}$$

$$f_{k\alpha} = \left(exp \left(\frac{E_{k\alpha}}{k_B T} \right) \right)^{-1} \quad . \tag{11}$$

We can calculate the gap parameter from eqn.4 by inverting the Bogoliubov transformation (eqn.6) and using eqns. (9-11) to calculate the operator averages. This procedure leads to the following self-consistent equation:

$$\left\langle a_{-k\beta} a_{k\alpha} \right\rangle = -\frac{\Delta_{k\alpha\beta}}{2E_{k\alpha}} \left(1 - 2f_{k\alpha} \right) \quad , \tag{12}$$

$$\Delta_{k\alpha\beta} = -\sum_{k'} \left\langle -k\alpha, k\beta | V | k^{'} \alpha^{'}, -k^{'} \beta^{'} \right\rangle \frac{\Delta_{k\alpha\beta}}{2E_{k\alpha}} tanh \left(\frac{E_{k\alpha}}{2k_B T} \right) \quad , \tag{13}$$

where I assumed that the interaction is diagonal in the spin indexes. If we further assume that the leading contribution to the interacting potential has L-wave symmetry and that it is non-vanishing close to the Fermi level only:

$$\langle -k\alpha, k\beta|V|k\alpha, -k\beta\rangle = \begin{cases} (2L+1)V_L P_L(\mathbf{k} \bullet \mathbf{k'}) & |\epsilon_k|, |\epsilon_{k'}| < \epsilon_c, \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

then eqn. 13 leads to:

$$\sum_k \left\langle \left(\Delta_k^+ \Delta_k\right)_{\alpha\alpha} \left[1 - |V_L| N(0) \int_0^{\epsilon_c} d\epsilon \frac{tanh(E_{k\alpha}/2k_B T)}{E_{k\alpha}}\right] \right\rangle_k = 0 \tag{15}$$

where $<>_k$ is the angular average over $\mathbf{k}$ and $N(0)$ is the density of states at the Fermi level.

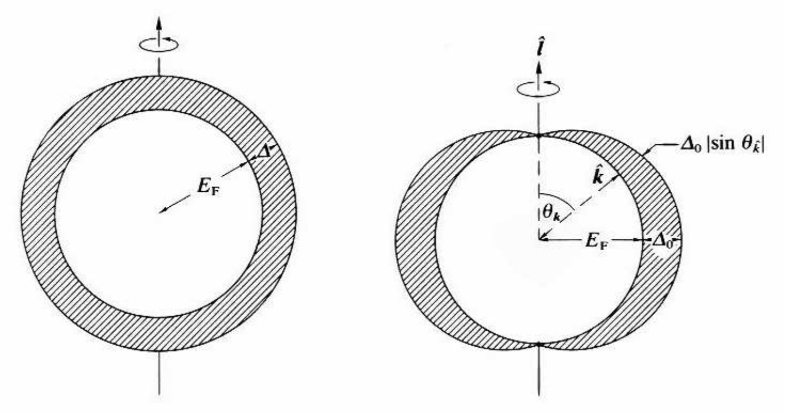

Figure 5: Schematic diagram of the energy gap in the a) BW state and b) ABM state.

In the case of the BW state, the 3 spin-triplet states $|\uparrow\uparrow>$, $(|\uparrow\downarrow> + |\downarrow\uparrow>)$ and $|\downarrow\downarrow>$ are populated, so the Δ_k matrix is unitary with 3 independent parameters, while in the case of the ABM state Δ_k is diagonal with spin independent elements because the population of the state with $S_z = 0$ is zero and there is no coupling between the other 2 states. As shown originally by Balian and Werthamer [6], the energy gap in the BW state is isotropic in space. Figure 5 shows a schematic diagram of the energy gap in the BW and ABM states. In the latter case the gap is anisotropic, vanishing along a specific direction, l. The square of the gap parameter can be shown to be:

$$\left(\Delta_k^+ \Delta_k\right)_{\alpha\beta} = \Delta_0^2 \left[1 - (k \cdot l)^2\right] \ . \tag{16}$$

Let's discuss the main thermodynamic properties of these two phases, as obtained from the mean field Hamiltonian, eqn. 8. If we assume that all the entropy is carried by the excited quasiparticles, a reasonable approximation since the pair condensed state is phase coherent, the entropy is just that of a gas of free particles with the usual form:

$$S = -k_B \sum_{k\alpha} \left[f_{k\alpha} ln\left(f_{k\alpha}\right) + \left(1 - f_{k\alpha}\right) ln\left(1 - f_{k\alpha}\right) \right] \; . \tag{17}$$

If we use eqn. 11, we can rewrite this as:

$$S = -\frac{1}{T} \sum_{k\alpha} \epsilon_k^2 \frac{\partial f_{k\alpha}}{\partial E_{k\alpha}} \; . \tag{18}$$

The free energy is obtained by taking the thermal average of eqn. 8 using eqns. 9-11, which reduces to:

$$F \equiv \langle H_{MF} - \mu N \rangle - TS = \frac{1}{2} \sum_{k\alpha} \left(\Delta_k^+ \Delta_k\right)_{\alpha\alpha} \left[\frac{1 - 2 f_{k\alpha}}{2 E_{k\alpha}} - \frac{1}{\epsilon_{k\alpha} + E_{k\alpha}} \right]$$

$$+ \sum_{k\alpha} E_{k\alpha} f_{k\alpha} + \sum_{k\alpha} \epsilon_{k\alpha}^2 \frac{\partial f_{k\alpha}}{\partial E_{k\alpha}} \tag{19}$$

$$F = \sum_{\epsilon<0} \epsilon_k - \frac{1}{4} N(0) \sum_{\alpha} \left\langle \left(\Delta_k^+ \Delta_k\right)_{\alpha\alpha} \right\rangle_k + \frac{1}{2} \sum_{k\alpha} \epsilon_{k\alpha}^2 \frac{\partial f_{k\alpha}}{\partial E_{k\alpha}} \quad , \tag{20}$$

where terms of order Δ/ϵ_c are neglected.

The specific heat can be calculated from the entropy:

$$C_V \equiv T \left(\frac{\partial S}{\partial T}\right)_V = -\frac{1}{T} \sum_{k\alpha} \frac{\partial f_{k\alpha}}{\partial E_{k\alpha}} \left[E_{k\alpha}^2 - \frac{1}{2} T \frac{\partial}{\partial T} \left(\Delta_k^+ \Delta_k\right)_{\alpha\alpha} \right] \; . \tag{21}$$

The second term gives rise to the discontinuity at T_c, which we can calculate analytically. Close to T_c we can write:

$$C_V = C_N + \Delta C_V + O(\Delta^2) \quad , \tag{22}$$

where C_N is the normal state specific heat and ΔC_V is the discontinous contribution. From eqn. 21 we get:

$$\Delta C_V = -\frac{1}{2} N(0) \frac{\partial}{\partial T} \sum_{\alpha} \left\langle \left(\Delta_k^+ \Delta_k\right)_{\alpha\alpha} \right\rangle_k \; . \tag{23}$$

Using the form of $\Delta(T)$ close to T_c found before for the two states, we get:

$$\frac{\Delta C_V}{C_N} = \begin{cases} \frac{12}{7\zeta(3)} \sim 1.43 & \text{BW state,} \\ \frac{10}{7\zeta(3)} \sim 1.19 & \text{ABM state.} \end{cases} \tag{24}$$

The low temperature expressions for C_V are given by:

$$C_V(T) = \begin{cases} 2\,(2\pi)^{1/2}\,k_B N(0)\Delta\left(\frac{\Delta}{k_B T}\right)^{3/2} e^{-\Delta/k_B T} & \text{BW state,} \\ \frac{7\pi^2}{5}\left(\frac{T}{\Delta_0}\right)^2 C_N(T) & \text{ABM state.} \end{cases} \tag{25}$$

The presence of the nodes of the energy gap in the ABM state significantly alters the low temperature properties of the system. Because of the nodes, quasi-particles can be excited along certain directions at arbitrarily small temperatures, contrary to the case of the BW state (and classic superconductors) where temperatures of the order of $T \sim \Delta/k_B T$ are needed to excite quasi-particles, leading to an exponential specific heat.

4 Conclusion

In this paper, I presented a short introduction to the superfluid phases of ^3He. ^3He has become one of the most important condensed matter systems and a detailed study of its properties is essential to future developments in the field. Many new phenomena was discovered during the study of this system, and new research areas emerged as a consequence.

References

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