Synchronization of Josephson Junctions

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Junseok Oh

Abstract: Synchronization is a common yet interesting emergent phenomenon observed in various parts of nature, such as condensed matter systems and biological coupled oscillators. How does a synchronization arise in a given system? This essay introduces a mathematical explanation to the emergence of synchronization and describes emergence of partially and completely synchronized phases in a series of current biased Josephson junctions.

Introduction

Collective synchronization is a phenomenon which individuals in the system spontaneously drive themselves into a common phase or frequency. The phenomenon of synchronization is very common in nature, from charge-density waves, phase locking of josephson junctions in series [1], cardiac pacemakers of heart cells, to fireflies blinking in sync [2][3], animals forming a small group (bird flocks and school of fish). One might think that it is so mysterious how individuals in these small systems communicate with each other to keep the harmony, for example how each firefly talk to the others in order to know when to blink. In fact, they do not; They are acting individually for their own sake, with only a weak interaction between them present. Simple synchronization can even be demonstrated with multiple metronomes on a moving platform where individual metronomes can interact through [4]. First out of phase with each other, they will drive themselves into a synchrony. This essay will introduce a mathematical model called Kuramoto model that can analytically explain the emergence of synchronization and related observations in many systems. Furthermore, synchronization in flashing of fireflies and Josephson junctions in series are studied as examples, with emphasis on different synchronized phases of Josephson junctions.

Kuramoto Model

Motivated by collective synchronization, Weiner and Winfree had tried to mathematically approach this universal behavior [5], but it was a Japanese physicist Yoshiki Kuramoto who came up with the famous Kuramoto model which is very useful in explaining the emergence of synchronization in many systems including the ones mentioned above. The detailed description of the model is described in [6]. The most general form for the coupled limit-cycle oscillators is:

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\theta_j - \theta_i), \qquad i = 1, \dots, N, \tag{1}$$

where θ_i, ω_i are the phase and natural frequency of i^{th} oscillator, Γ is the interaction function that can be calculated from limit-cycle model. Kuramoto set the interaction function to be sinusoidal with assumptions that the distribution of the frequencies is unimodal and symmetric about its center (like Gaussian):

Fig 1. Synchronous flashing of fireflies. Captured from Youtube video. [3]

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \qquad i = 1, \dots, N, \tag{2}$$

where K is a positive coupling strength, and ω_i is now a deviation from the mean frequency Ω [6]. Next, take complex order parameter that describe the overall wavefunction

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} \tag{3}$$

where r is phase coherence and ψ is average phase. For example, r = 1 means all the oscillators in the system are acting as one coherent oscillator, where r = 0 means they are not coupled at all. Kuramoto realized that multiplying $e^{-i\theta_i}$ to both sides gives

$$re^{i(\psi-\theta_i)} = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j-\theta_i)} \tag{4}$$

and take the imaginary part,

$$r\sin(\psi - \theta_i) = \frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) \tag{5}$$

so we can write the Kuramoto equation as:

$$\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i) \tag{6}$$

Through this approach, Kuramoto simplified the equation so that individual oscillators are seemingly uncoupled and only interact via mean field frequency ψ and coherence strength r.

Fireflies – Emergence of Synchronization

Using Kuramoto model as in form of Eqn (6), we can apply it to understand to a certain degree the synchronization in fireflies' flashing, which can be observed in some parts of Southeast Asia [2]. However, the same model can be used to describe similarly synchronized systems, step by step derivation shown in [7]. Assume each firefly flash when $\theta = 0$. Writing $\phi = \psi - \theta$ and noticing $\dot{\psi} = \Omega$,

$$\dot{\phi} = \Omega - \dot{\theta} = (\Omega - \omega) - A\sin(\phi) \tag{7}$$

Notice here there are no indices, because the oscillators are uncoupled. Further nondimensionalizing the equation by defining

$$\tau = Krt, \qquad \mu = (\Omega - \omega)/Kr \tag{8}$$

we get

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \mu - \sin(\phi) \tag{9}$$

From this, we can think of three cases: $\mu = 0,\ 0 < \mu < 1,\ \mu > 1.$

The phase diagrams for the three cases are shown in Fig. 1. When $\mu = 0$ $(\Omega = \omega)$, the stable fixed point is located at $\phi *= 0$. For all states with $\phi < 0$ has $\phi' > 0$ and states with $\phi > 0$ has $\phi' < 0$, which means that in long term, all states are attracted towards the stable fixed point $\phi *= 0$. All oscillators will be oscillating at their natural frequency ω. When $0 < \mu < 1$, the stable point $\phi *$ is shifted away from 0, but again, the oscillators will eventually be phase locked to this $\phi *$. However, for $\mu > 1$, there is no zero point crossing as seen in the diagram, and thus no fixed point at all. An oscillator would therefore would not stabilize at one particular frequency, but drift away from Ω. From this simple analysis, we can see that for $|\mu| < 1$, i.e. for $\omega - Kr < \Omega < \omega + Kr$, fireflies will flash in unison, and this range of frequency is called the range of entrainment [7].

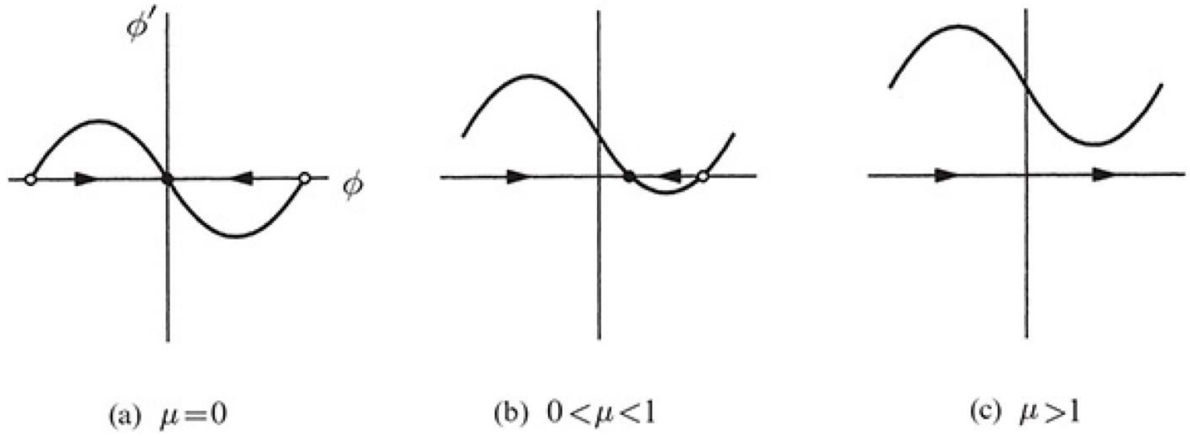

Fig. 2) Phase diagram for three cases for different values of μ [7]

We looked at how synchronization in flashing of fireflies emerge by utilizing Kuramoto model. The same analysis can be done for similar systems, where individual oscillators are entrained to the stimuli from the surroundings and synchronize. This is very much simplified version of the work done by Ermentrout in 1991 [8]

Josephson Junction

Josephson effect, discovered by British physicist Brian David Josephson, is a phenomenon where a supercurrent flows without any voltage applied across a superconducting tunnel junction, known as a Josephson junction. For the discovery of this effect, Josephson was awarded Nobel prize in physics in 1973. The basic idea of the effect is that superconducting Cooper pairs tunnel through the junction. The effect is very widely applied in many different fields, inside and outside physics. The two governing equations for a Josephson junction are

$$\begin{aligned} V(t) &= \frac{\hbar}{2e}\dot{\phi} \\ I(t) &= I_c \sin\phi \end{aligned} \tag{10}$$

Here, V is the voltage, I is the supercurrent, I_c is the critical current, and ϕ is the phase difference across the junction [7]. More realistically, a electric circuit with a Josephson junction will have resistance R, inductance L, and capacitance C within it. We will disregard L for now. Then we can draw a circuit as Fig. 4. Kirchoff's law along with the above equations give us

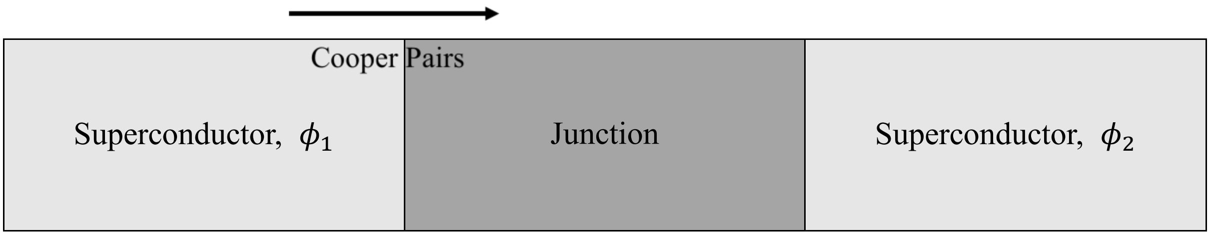

Fig. 4) A Josephson Junction

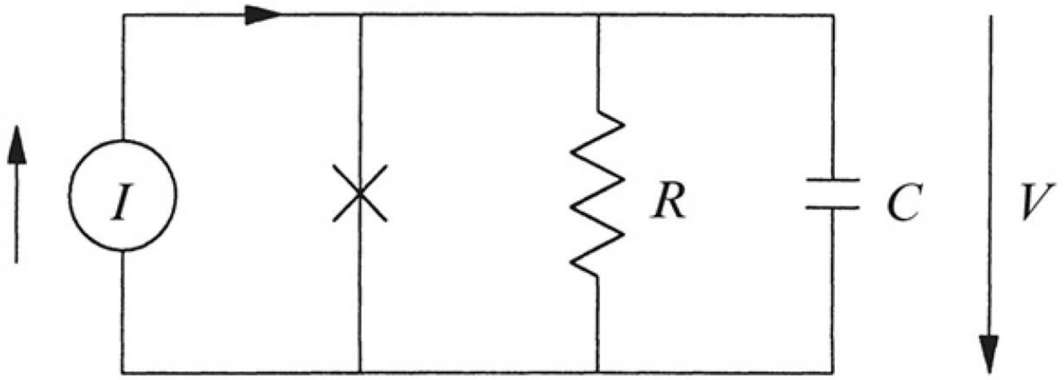

Fig. 3) Realistic Josephson junction circuit. Josephson junction is denoted by 'X' sign [7]

$$\frac{\hbar C}{2e}\ddot{\phi} + \frac{\hbar}{2eR}\dot{\phi} + I_c \sin\phi = I \tag{11}$$

A direct analogy to a mechanical pendulum driven with a constant torque [7]:

$$mL^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = \Gamma \tag{12}$$

Series Array of Josephson Junctions

The behavior of series of Josephson junctions is very interesting and not fully understood. Some of the interesting physics of the system regarding Floquet multipliers [9] and neutral stability [10] in splay-phase states, and synchronization phase transitions [1] has been studied by series of papers written by Strogatz and others. In particular, we will look closely at the system phase transition into different synchronization states in the disordered Josephson series arrays studied by Wiesenfeld, Colet, and Strogatz. As before, we can derive our governing equations from Kirchoff's law for current and voltage.

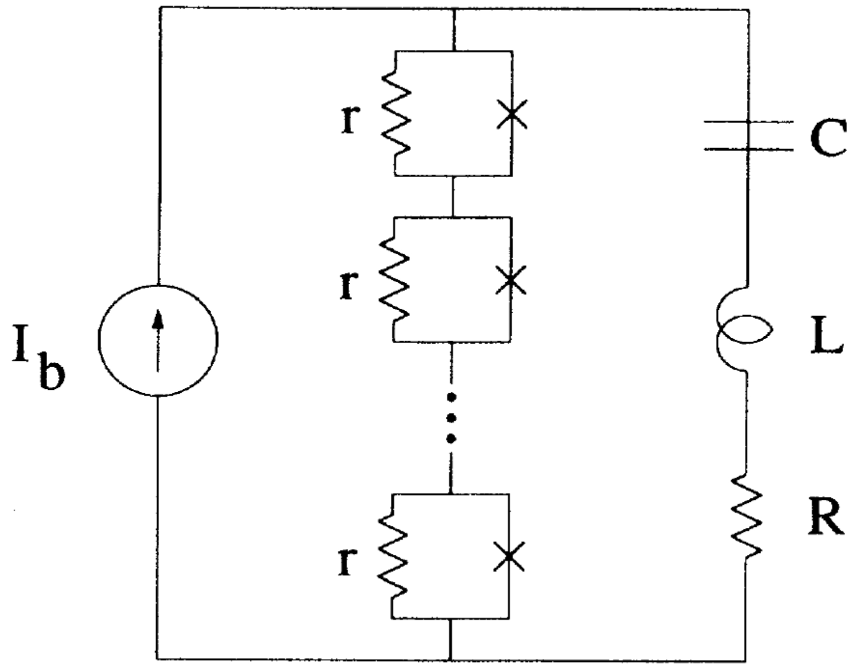

Fig. 5) Circuit schematic of series array of Josephson junctions [9]

$$\frac{\hbar}{2er_j}\dot{\phi}+I_j\sin\phi_j+\dot{Q}=I_B,\qquad L\ddot{Q}+R\dot{Q}+\frac{1}{C}Q=\frac{\hbar}{2e}\sum_{k=1}^{N}\dot{\phi}_k,\qquad j=1,\dots,N \qquad (13)$$

where I_B is bias current, I_j and r_j are junction critical current and resistance of the jth junction, Q is the charge on the capacitor, and the other parameters are same as before. [1] Define now natural angle θ as $\frac{d\theta_j}{\omega_j}=\frac{d\phi_j}{(2er_j/\hbar)(I_B-I_j\sin\phi_j)}$ where θ is natural in the sense $\dot{\theta}$ is constant in the limit of weak coupling. Using trigonometric relation $I_B-I_j\,sin\,\phi_j=({I_B}^2-{I_j}^2)/(I_B-I_j\,cos\,\theta_j)$, the equation (13) becomes

$$\dot{\theta_j}=\omega_j-\frac{\omega_j\dot{Q}}{{I_B}^2-{I_j}^2}(I_B-I_j\cos\theta_j) \qquad (14)$$

It is shown in a paper by Weisenfeld and Swift [11] that the RHS of the above equation can be time averaged to first order to produce

$$\dot{\theta_j}=\omega_j-\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\theta_k+\alpha) \qquad (15)$$

where

$$K = \frac{Nr\omega(2erI_B/\hbar - \omega)}{[(L\omega^2 - C^{-1})^2 + \omega^2(R + Nr)^2]^{1/2}}$$
$$\cos\alpha = \frac{L\omega^2 - C^{-1}}{[(L\omega^2 - C^{-1})^2 + \omega^2(R + Nr)^2]^{1/2}} \tag{16}$$

Notice that Eqn (15) is the familiar Kuramoto model introduced earlier, and we successfully mapped the system onto Kuramoto model. The technique using the imaginary part of the complex order parameter $\sigma e^{i\psi}$ is used here again, with step by step derivation by Sakaguchi and Kuramoto. [12] The paper also shows the frequency Ω and the order parameter magnitude σ can be determined. The range of entrainment is $\omega - K\sigma < \Omega < \omega + K\sigma$, same as in the case of fireflies. Call the unimodal, symmetric frequency distribution $g(\omega)$, then we can calculate the fraction of the oscillators that are phase locked by integrating the distribution over the range of entrainment:

$$f = \int_{\Omega-K\sigma}^{\Omega+K\sigma} d\omega\, g(\omega) \tag{17}$$

With some reasonable values for typically Josephson junctions, the paper also compares plot from the analytic results and the one from numerically calculation using Eqns (13), as shown in Fig. 6. The numerical calculation assumes normalized parabolic distribution of critical currents of junctions to be

$$P(I) = \frac{3}{4\Delta^3}[\Delta^2 - (I - \bar{I})^2] \tag{18}$$

with full width 2Δ and average critical current $\bar{I}$.

Plots in Fig. 6 and 7 suggest very interesting fact. For $\Delta > \Delta_C \approx 0.13$, f is close to 0, but as Δ crosses Δ_C, phase-locking begins to happen, and the system undergoes transition into partially phase-locked state. Eventually when Δ decreases below $\Delta_L \approx 0.02$, f reaches 1 and the system is completely phase-locked. Points (a), (b), and (c) each represents phases of different degree of synchronization. The power spectra for the whole system shown in Fig. 7. At (c), the system is at the onset of transition from completely uncoupled to partially synchronized regime, so the peak at $\omega \approx 2.2$ starts to appear in the plot. As the system goes into (a), completely synchronized regime, only the peak survives and the low frequency noises quench.

Now shown that there are three distinct phases of synchronization in series arrays of Josephson junctions, the authors give what to look for to experimentally observe this emergent behavior. One of their considerations is that Δ, the degree of disorder is not the most experimentally practical parameter to vary to observe these phases. Rather, the realistic control parameter would be I_B, the bias current applied. Also, instead of directly measuring the number or the fraction of the synchronized oscillators, experimenters will be measuring the amplitude of the peak as shown in Fig. 7, given as:

$$A_\Omega = 2\mathrm{K}\sigma \frac{{I_B}^2 - \bar{I}^2}{\bar{\omega}^2 \bar{I}} \sqrt{(L\Omega^2 - C^{-1})^2 + \Omega^2 R^2} \tag{19}$$

which is proportional to the order parameter magnitude, σ. Thus, A_Ω is a good order parameter for determining the transition between the phases. As seen in Fig. 7, the onset of ordering (from phase (c) to (b)) is clear since the peak emerges. However, transition into the completely locked phase is not as clear, the only signal being quenching of low frequency noises near the line. To simulate the experiment, the authors plot A_Ω and along with the fraction of phase locked junctions versus I_B, acquired from numerical simulations from Eqns (13) and analytic results for the fraction of phase-locked junctions from Eqn (17) and strength of the line in Eqn (19).

As expected, compared with the fraction, A_Ω clearly signals the transition into partially synchronized regime at around $I_B = 4.3\text{mA}$ for $\Delta = 0.001\text{mA}$. However, transition into fully synchronized phase is not so clear as previously predicted. Further lowering I_B, the system loses its dynamical stability. The same is with $\Delta = 0.002\text{mA}$, except the system is never fully synchronized. Overall, the plot suggests that the transition at the birth of the order in the system can be experimentally fully observed looking at the response of the system power spectra to I_B, and the same for the transition to complete synchronization to some degree.

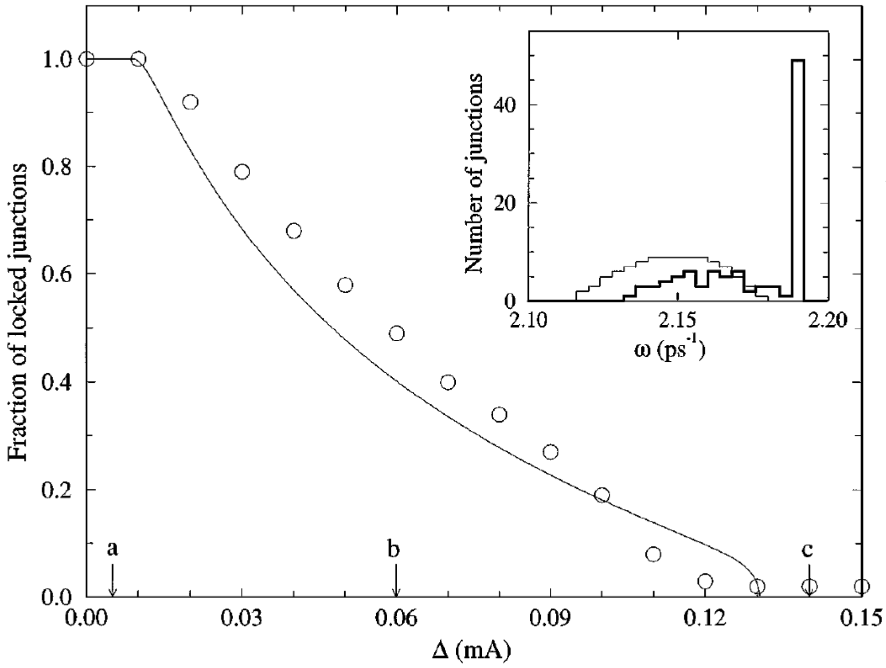

Fig. 6) Fraction of phase locked junctions with typical parameters for Josephson junctions. Circles are from the numerical calculations and solid line is analytic result, Eqn (17). Inset: bare frequency (thin) and dressed frequency (thick) distributions at (b) $\Delta = 0.06$ [1].

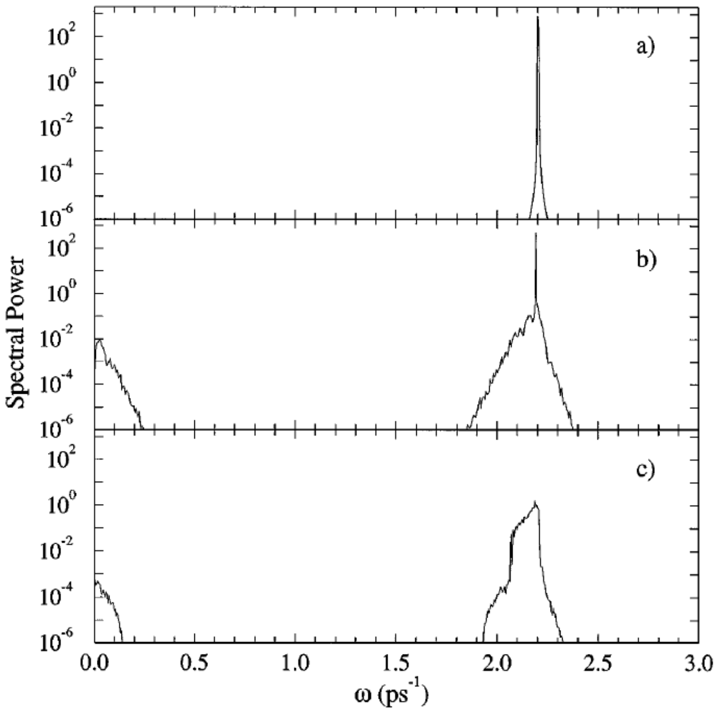

Fig. 7) Power spectra for the whole array at different values of Δ, given by $(\hbar/2\mathrm{e})(\sum \dot{\phi}_k - \langle \sum \dot{\phi}_k \rangle)$. A peak around $\omega \approx 2.2$ grows as the system goes from (c) to (a) phase in Fig. 6 [1].

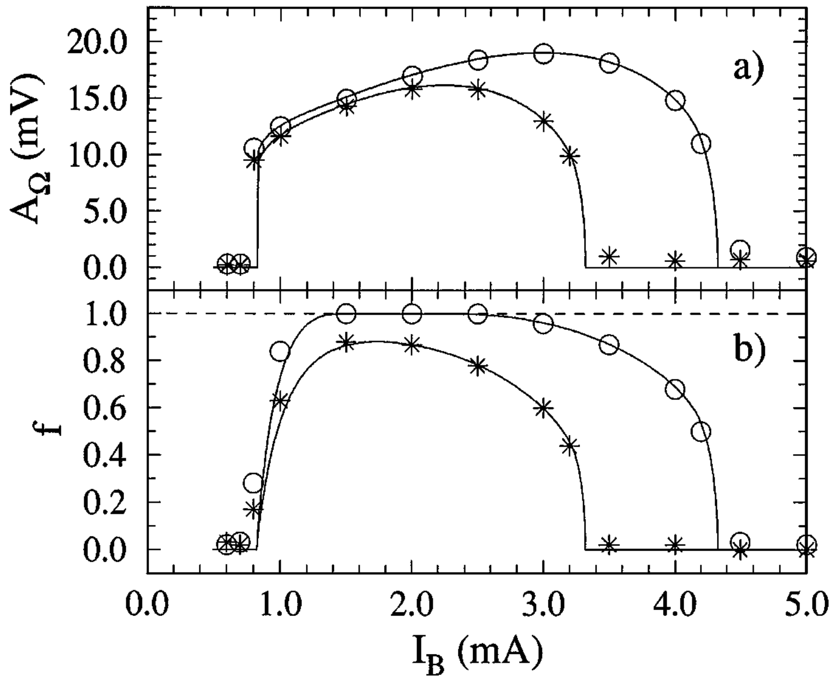

Fig. 6) Dependence of A_Ω (a) and f (b) on I_B at $\Delta = 0.001\text{mA}$ (circle) and $\Delta = 0.002\text{mA}$ (asterisks) based on Kirchoff Eqns (13). Solid line represents analytic Eqn (17) and (19) [1]

Conclusion

The essay started addressing general idea of synchronization, explaining its emergence using Kuramoto model in a simple example of fireflies. In the end, we looked at the synchronization phases and transitions in series array of Josephson junctions. The authors of the paper analyzed the series of Josephson junctions to show three different synchronization states of the system utilizing Kuramoto model. Numerical simulations based on the basic Kirchoff's law were shown to match with the mathematical model. Furthermore, they provide a way to experimental observe this event by looking at the amplitude of the line in power spectrum. I was able to find a few papers that claims to have experimentally observed the synchronization in the system but was not obvious if the experimenters used the methods suggested by the authors of the paper presented here.

People with backgrounds in physics might see the emergence of synchronization as just the result of coupled oscillators resonating at a given frequency. However, I found that the field of nonlinear dynamics and study of synchronization in specific is more interesting than I had suspected, in the fact that seemingly collective behaviors naturally emerge in complex systems of school of animals, Josephson junction arrays, and others and that there exists a simple model most of these systems can be mapped onto.

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