Magnons: Spin-Waves in Magnetic Materials

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The purpose of this essay is to develop the notion of emergent magnon excitations which appear in magnetic materials. In particular, we will discuss the so-called Holstein-Primakoff boson and Schwinger boson representations of spin-operators. This technique is powerful in that we can utilize the second quantization formalism discussed in class; however, we will find this approach places a constraint on the Hilbert space of the operators.[1] With the preceding formalism established, we will apply it to the Heisenberg model (a minimal model of a ferromagnet) and show that the low-energy quasiparticle excitations describing such a system are spin-waves.[2] We will conclude with a discussion of potential technologies which can exploit spin-wave physics — a field known as magnonics.[3]

1 A Minimal Model of a Magnet

In this paper we will discuss the low-energy physics of a large class of magnetic materials; our goal is to ultimately determine the bulk properties of such systems by tracing over the microscopic degrees of freedom. For our purposes, a “magnet" is a collection of spins embedded on a lattice in which the spins can interact with external magnetic fields and with one another.

To begin, recall that the Hamiltonian of a spin S in an external magnetic field B_0 can be written both classically and quantum mechanically as

$$H = -B_0 \cdot m \tag{1}$$

where $m = \alpha S$ is the corresponding magnetic moment with α being constant (e.g. for an electron $\alpha = g\mu_B/\hbar$ where g is the *g-factor* and μ_B the Bohr magneton).[4] [Note: we adopt units in which $\alpha = 1$ so that $H = -B_0 \cdot S$ and also set $\hbar = 1$.]

Various models can be been used to capture different aspects of magnetism; in this paper we intend to discuss properties of the *ferromagnetic* phase which is exhibited in many systems; we choose to employ a minimal model of a magnet known as the *Heisenberg model.* The Hamiltonian is

$$H = -\sum_x B_0 \cdot S_x - \frac{1}{2}\sum_{x,x'} J_{xx'} S_x \cdot S_{x'} \tag{2}$$

where $S_x = (S_x^1, S_x^2, \ldots, S_x^n)$ is the spin at position $x = (x^1, x^2, \ldots, x^D)$, B_0 is a (uniform) external magnetic field felt by each spin, and $J_{xx'}$ represents the strength of the coupling between the spin at x and x'; n is the embedding dimension and D the number of spatial dimensions.

For simplicity let's place the system on a hypercube (with lattice spacing a) so that a site has coordinates $x^\mu = am^\mu$ where $m^\mu = 1, 2, \ldots, N$ (and $\mu = 1, 2, \ldots, D$); the total number of sites is then N^D. Furthermore, let's consider a very large lattice so that it is appropriate to employ periodic boundary conditions — wave vectors in the first Brillouin zone then have the form $k^\mu = \frac{2\pi}{Na} n^\mu$ where $n^\mu \in \mathbb{Z}$ and span the values $n^\mu = (-N/2, +N/2]$.

From Eq. 2 we see that when $J_{xx'} > 0$ the interaction is ferromagnetic and when $J_{xx'} < 0$ it is antiferromagnetic (we will remain in the ferromagnetic regime). Furthermore, the interaction term possesses $O(n)$ symmetry under global rotations of the spins. This symmetry is broken (by hand) due to the presence of the external magnetic field though.

We wish to determine bulk properties of a conventional magnet and hence choose to work with $D = n = 3$; furthermore, we rotate coordinates so that the external magnetic field is aligned along the x^3-axis.

2 Representations of Spin Operators

We wish to treat our model quantum mechanically; in this case, the spin operators will be the generators of $SU(2)$ which satisfy the algebra

$$[S^i, S^j] = i\epsilon^{ijk} S^k \tag{3}$$

where $i = 1, 2, 3$ and ϵ is the Levi-Civita tensor. [Note: the dimension of the group (which is 3) places a constraint of the value of the embedding dimension; a constraint not present classically.] Different representations of the group correspond to different values of spin — in this section, consider the general case of spin-s; here the generators are $(2s+1)$ by $(2s+1)$ matrices with eigenvalues running from $-s, -s+1, \ldots, s-1, s$. It is conventional to work in a basis in which S^3 is diagonal and we denote its eigenvectors by $S^3 |m\rangle = m |m\rangle$ where m runs over the eigenvalues.

Our goal in the rest of this section is to take a detour to discuss representations of spin-operators.[1]

2.1 Holstein-Primakoff Bosons

We start with the spin operator $S = (S^1, S^2, S^3)$ and define the raising and lowering operators in the usual way as

$$S^{\pm} = S^1 \pm iS^2 \tag{4}$$

Now, we introduce the following transformation due to Holstein and Primakoff:

$$S^+ = \sqrt{2s - b^\dagger b}\; b \tag{5}$$

$$S^- = b^\dagger \sqrt{2s - b^\dagger b} \tag{6}$$

$$S^3 = s - b^\dagger b \tag{7}$$

where b are the *Holstein-Primakoff bosons* obeying $[b, b^\dagger] = 1$ with $b^\dagger b |n_b\rangle = n_b |n_b\rangle$.[1] Immediately one sees that there is something unphysical about this transformation; namely, we know the set of eigenstates of S^3 are $\{|s\rangle, \cdots, |-s\rangle\}$, but Eq. 7 implies we have the set of eigenstates $\{|s\rangle, \cdots, |-s\rangle, |-s-1\rangle, \ldots, |-\infty\rangle\}$! The Hilbert space spanned by the Holstein-Primakoff bosons is too large — thus, in order to employ this spin operator representation, one must project onto the physical states by enforcing the constraint $n_b \leq 2s$.[1]

As a sanity check, we should convince ourselves that Eq.s 5-7 do indeed satisfy the $SU(2)$ algebra of Eq. 3. To do this, begin by inverting Eq. 4 to get $S^1 = (S^+ + S^-)/2$ and $S^2 = (S^+ - S^-)/2i$. Then it follows that

$$[S^1, S^2] = \frac{i}{2}[S^+, S^-] = \frac{i}{2}[\sqrt{2s - b^\dagger b}\; b, b^\dagger \sqrt{2s - b^\dagger b}] \tag{8}$$

Using the commutator algebra of the bosons and employing the constraint on the Hilbert space $(2s - b^\dagger b \geq 0)$ one can compute the commutator to obtain

$$[S^1, S^2] = i(s - b^\dagger b) = iS^3 \tag{9}$$

which agrees with Eq. 3 (the other commutators are computed in the same way). We should also compute the operator $(S)^2 = S^i S^i$. Provided the constraint on the Hilbert space is enforced, this takes a particular nice form in the Holstein-Primakoff representation: $(S)^2 = s(s+1)$; namely, the spin angular momentum is simply given by its eigenvalue.[1]

2.2 Schwinger Bosons

Another bosonic representation of the spin operators is given by

$$S^+ = a^\dagger b \tag{10}$$

$$S^- = b^\dagger a \tag{11}$$

$$S^3 = \frac{1}{2}(a^\dagger a - b^\dagger b) \tag{12}$$

where a and b are the *Schwinger bosons* obeying the usual commutation relations; we denote an eigenstate of their number operators by $|n_a, n_b\rangle$.[1] It is straightforward to show that these operators satisfy the $SU(2)$ algebra. Notice that once again though we run into an issue with the Hilbert space of the operators as there are unphysical eigenvalues of S^3 (they can take on any integer or half-integer value!). It is instructive to look at the operator $S^i S^i$, after some algebra one finds that

$$(S)^2 = \frac{1}{2}(a^\dagger a + b^\dagger b)\left[\frac{1}{2}(a^\dagger a + b^\dagger b) + 1\right] \tag{13}$$

which has eigenvalues $\frac{1}{2}(n_a + n_b)[\frac{1}{2}(n_a + n_b) + 1]$, but physical states must have the eigenvalue $s(s+1)$. Therefore physical states correspond to the case $n_a + n_b - 2s = 0$, this is the constraint we must enforce.[1]

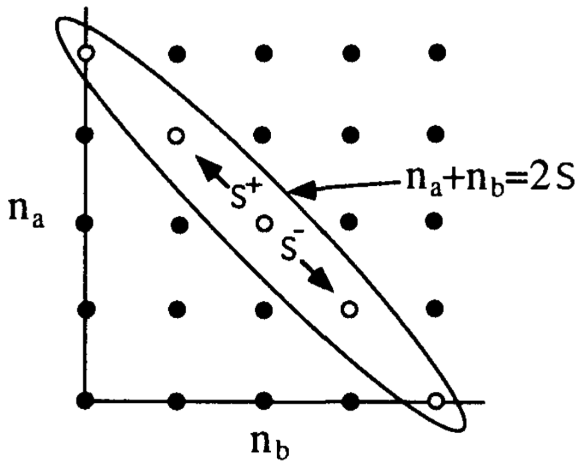

Figure 1: The eigenvalue spectrum of the schwinger bosons as shown in [1]. The diagonal line highlights those states which are physical.

We have now seen two spin representations built using bosonic ladder operators. Both representations prove useful when carrying out computations in different phases of the Heisenberg model (we will employ Holstein-Primakoff bosons shortly).[1] Their worth owing to the fact that we can utilize straightforward commutator algebra, but this comes at the price of enlarging the operators' Hilbert space. In general, we enforce the constraint that we remain in the physical subspace by introducing a projection operator P_S: in the Schwinger boson case, this operator would need to satisfy $P_S(a^\dagger a + b^\dagger b - 2s) = 0$.[1] A powerful alternative approach to treat such constraints is through the introduction of Lagrange multipliers in the partition function.[5]

3 Magnons: Spin-Wave Excitations

Let us now return to the Heisenberg model and apply the results of the previous section to spin$-1/2$ particles. Recalling that the external field is orientated to point along the (positive) x^3 axis, the model is

$$H = -B_0 \sum_x S_x^3 - \frac{1}{2} \sum_{x,x'} J_{xx'} S_x^i S_{x'}^i \tag{14}$$

To proceed, represent the spin operators using Holstein-Primakoff bosons:

$$S_x^+ = \sqrt{1 - b_x^\dagger b_x}\; b_x \tag{15}$$

$$S_x^- = b_x^\dagger \sqrt{1 - b_x^\dagger b_x} \tag{16}$$

$$S_x^3 = \frac{1}{2} - b_x^\dagger b_x \tag{17}$$

where $[b_x, b_{x'}^\dagger] = \delta_{xx'}$.[2] The linear term in the Hamiltonian is simple as it is diagonal in the operators:

$$\sum_x S_x^3 = \frac{1}{2} N^3 - \sum_x b_x^\dagger b_x \tag{18}$$

where recall that N^3 is the total number of sites on the lattice. The quadratic term is bit more involved; one has

$$\begin{aligned} S_x^i S_{x'}^i &= S_x^1 S_{x'}^1 + S_x^2 S_{x'}^2 + S_x^3 S_{x'}^3 && (19) \\ &= \frac{1}{2}\left(S_x^+ S_{x'}^- + S_x^- S_{x'}^+\right) + S_x^3 S_{x'}^3 && (20) \\ &= \frac{1}{2}\left[\sqrt{1 - b_x^\dagger b_x}\; (b_x b_{x'}^\dagger)\sqrt{1 - b_{x'}^\dagger b_{x'}} + b_x^\dagger \sqrt{1 - b_x^\dagger b_x}\sqrt{1 - b_{x'}^\dagger b_{x'}}\; b_{x'}\right] \\ &+ \left(\frac{1}{2} - b_x^\dagger b_x\right)\left(\frac{1}{2} - b_{x'}^\dagger b_{x'}\right) && (21) \end{aligned}$$

One is then able to expand the square root in a power series of $b_x^\dagger b_x$.[1] Working to quadratic order in the operators gives us the interaction

$$S_x^i S_{x'}^i = \frac{1}{2}\left[b_x b_{x'}^\dagger + b_x^\dagger b_{x'} - b_x^\dagger b_x - b_{x'}^\dagger b_{x'} + \frac{1}{2}\right] + \mathcal{O}(b_x^4) \tag{22}$$

What we have just done is crucial to the physics we hope to probe and warrants further discussion. First, recall from our earlier discussion that the validity of the Holstein-Primakoff transformation requires we project onto the states which obey $P_S(1 - b_x^\dagger b_x) \geq 0$ where we set $s = 1/2$. Let us also restate our intention is to probe the ferromagnetic phase; namely, the phase in which most of the spins are aligned along the same direction. At low-temperatures the spins will tend to align along the (positive) x^3-axis due to the external field which then implies the magnetization should be [2]

$$M^3 = \sum_x \langle S_x^3 \rangle = \frac{1}{2} N^3 - \sum_x \langle b_x^\dagger b_x \rangle \approx \frac{1}{2} N^3 \tag{23}$$

where $\langle \dots \rangle$ is a thermal expectation value. Namely at low-energies, when the system is in the ferromagnetic phase, we have $\langle b_x^\dagger b_x \rangle \ll 1$. Therefore, on the average, the constraint on the Holstein-Primakoff bosons is indeed satisfied! The constraint fails to be satisfied at high enough temperatures where $\langle b_x^\dagger b_x \rangle \approx 1$; thus, as a measure of the regime of applicability of our theory, we define a temperature scale T_{HP} at which $\langle b_x^\dagger b_x \rangle_{T_{HP}} = 1$.

Furthermore, as the temperature increases thermal fluctuations will cause the spins to change orientation. Eventually we reach a critical temperature denoted T_c (the so-called *Curie temperature*) in which a phase transition occurs and we leave the ferromagnetic phase; this is signified by the order parameter M^3 vanishing.[2] However, in this paper we are only interested in physics far away from this region of the phase diagram ($T \ll T_c$); thus, we return to the Heisenberg model and insert Eq.s 18 and 22 to find

$$H = -\frac{N^3}{2}B_0 - \frac{1}{8}\sum_{xx'} J_{xx'} + \sum_x \left(B_0 + \frac{1}{2}\sum_{x'} J_{xx'} \right) b_x^\dagger b_x - \frac{1}{4}\sum_{xx'} J_{xx'}(b_x b_{x'}^\dagger + b_x^\dagger b_{x'}) \tag{24}$$

where we dropped all terms of $\mathcal{O}(b_x^4)$ and used that $J_{xx'}$ is symmetric in its indices. The Hamiltonian is not diagonal in the Holstein-Primakoff bosons; however, its structure is simple in that it only consists of free and hoping bosons. Therefore we expand the operators (which create spin excitations) in plane-waves or *spin-waves*.[2]

We define a **magnon** excitation with momentum k by the expansion

$$b_k^\dagger \equiv \frac{1}{N^{3/2}}\sum_x e^{+ik\cdot x} b_x^\dagger \qquad , \qquad b_k = \frac{1}{N^{3/2}}\sum_x e^{-ik\cdot x} b_x \tag{25}$$

where $k\cdot x = k^\mu x^\mu$ with $k^\mu = (-\frac{\pi}{a}, +\frac{\pi}{a}]$ (recall we use periodic boundary conditions).[2] It is a quick exercise to demonstrate that magnons are bosonic quasiparticles obeying commutation relations: $[b_k, b_{k'}] = 0$ and $[b_k, b_{k'}^\dagger] = \delta_{kk'}$.

In $D = 3$ we can write the Kronecker delta as $\delta_{kk'} = \frac{1}{N^3}\sum_x \exp[i(k-k')\cdot x]$ which allows us to invert the transformation; namely, a Holstein-Primakoff boson can be written as a linear combination of spin-waves via

$$b_x^\dagger = \frac{1}{N^{3/2}}\sum_k e^{-ik\cdot x} b_k^\dagger \qquad , \qquad b_x = \frac{1}{N^{3/2}}\sum_k e^{+ik\cdot x} b_k \tag{26}$$

Let us now write Hamiltonian in terms of the magnon creation/destruction operators by directly inserting Eq.s 26 into Eq. 24. As a preliminary though, focus on the coupling $J_{xx'}$ — it's symmetric in its indices and must be a periodic function due to the boundary conditions; let us further consider a system that possesses *translation invariance* which implies $J_{xx'} = J(x-x')$. Because of this, it proves useful to Fourier transform the coupling:[2]

$$\tilde{J}_k \equiv \sum_x J_{x0} e^{-ik\cdot x} \qquad \longleftrightarrow \qquad J_{xx'} = \frac{1}{N^3}\sum_k \tilde{J}_k e^{+ik\cdot(x-x')} \tag{27}$$

Armed with Eq. 26 and 27, the terms in the Hamiltonian now become

$$-\frac{1}{8}\sum_{xx'} J_{xx'} = -\frac{N^3}{8}\tilde{J}_0 \tag{28}$$

$$\sum_x \left(B_0 + \frac{1}{2}\sum_{x'} J_{xx'}\right) b_x^\dagger b_x = \sum_k \left(B_0 + \frac{1}{2}\tilde{J}_0\right) b_k^\dagger b_k \tag{29}$$

$$-\frac{1}{4}\sum_{xx'} J_{xx'}(b_x b_{x'}^\dagger + b_x^\dagger b_{x'}) = -\frac{1}{4}\sum_k \tilde{J}_k - \frac{1}{2}\sum_k \tilde{J}_k b_k^\dagger b_k \tag{30}$$

After all this work, we finally see that at low-temperatures in the ferromagnetic phase of the Heisenberg model, the effective Hamiltonian is diagonal in the magnon quasiparticle operators:[2]

$$H = E_0 + \sum_k \omega_k b_k^\dagger b_k \tag{31}$$

where (after some use of Eq. 27) we have defined

$$E_0 \equiv -\frac{N^3}{2}\left(B_0 + \frac{1}{2}J_{00} + \frac{1}{4}\sum_x J_{x0}\right) \tag{32}$$

$$\omega_k \equiv B_0 + \frac{1}{2}\left(\tilde{J}_0 - \tilde{J}_k\right) = B_0 + \sum_x J_{x0}\sin^2\left(\frac{k\cdot x}{2}\right) \tag{33}$$

Notice that because we are only working in the ferromagnetic regime of the model where $J_{xx'} \geq 0$ and because $B_0 \geq 0$, the dispersion relation ω_k will always be non-negative. Therefore, E_0 is truly the ground state energy of the model. Furthermore, notice that if we turn off the coupling between spins ($J_{xx'} = 0$) the ground state energy Eq. 32 correctly reproduces the ground state energy of N^3 free spins in an external magnetic field.

Therefore, we have discovered that the emergent description of a ferromagnet at low temperatures is that of free spin-wave bosons with dispersion relation ω_k. Starting with a lattice of quantum spins we have seen two levels of emergence: first, the effective description in terms of (spin-localized) Holstein-Primakoff bosons; then this naturally led to a description of the system in terms of (spin-wave) magnons.

As a final note on the effective model, let us get a feel for the dispersion relation in a simple scenario. Consider the case of nearest-neighbour interactions; in this case, $J_{xx'} = J$ for nearest neighbours, but is otherwise zero. One finds that the dispersion for momentum $k = (k^1, k^2, k^3)$ is

$$\omega_k = B_0 + J\left[3 - \cos\left(k^1 a\right) - \cos\left(k^2 a\right) - \cos\left(k^3 a\right)\right] \tag{34}$$

A plot of the dispersion relation along the k^1 momentum axis is included below. One can see that the low-energy behaviour is parabolic $\omega_k \sim (k^1)^2$ and at higher energy there is a region in which the dispersion is approximately linear $\omega_k \sim |k^1|$; these look like gas and phonon branches respectively. However, one must be very careful applying these notions, because the dispersion depends on all k^μ values and we singled out a particular axis to plot it. Nonetheless, in the case that (k^1, k^2, k^3) are all small we can Taylor expand the cosines to obtain[2]

$$\omega_k \approx B_0 + \frac{Ja^2}{2}|k|^2 \tag{35}$$

Therefore, we can think of the region of k-space in which $|ka| \ll 1$ as a *magnon gas.*

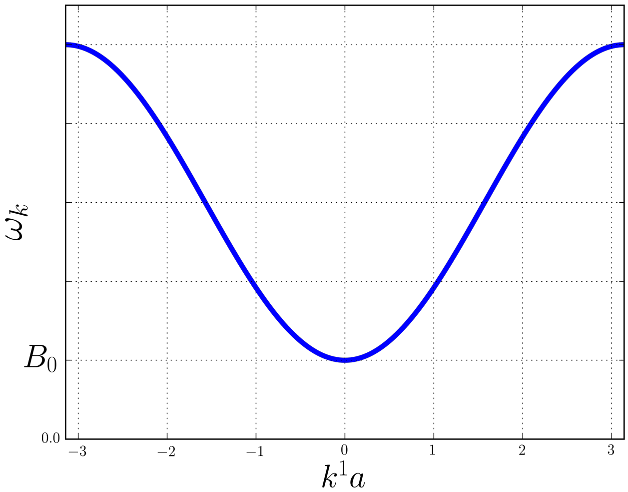

Figure 2: Magnon dispersion relation plotted against momenta $(k^1, 0, 0)$ for a nearest-neighbour interaction of strength J.

4 Thermodynamics of Magnons

Starting with the spin-wave Hamiltonian of Eq. 31, we can easily determine the partition function because the collective magnon excitations are non-interacting. In the grand-canonical ensemble we have

$$Z = \operatorname{tr} e^{-\beta(H-\mu N)} = e^{-\beta E_0} \prod_k \frac{1}{1 - e^{-\beta(\omega_k - \mu)}} \tag{36}$$

The thermal average number of magnons with momentum k is then given by the Bose-Einstein distribution function:

$$\langle b_k^\dagger b_k \rangle = \frac{1}{e^{\beta(\omega_k - \mu)} - 1} \tag{37}$$

We are now in a position to compute the (physically observable) magnetization of the system $M = (M^1, M^2, M^3) \equiv \sum_x \langle S \rangle$. Specifically, consider the component of the magnetization along the axis of the external field (we chose the x^3 axis); from our previous results we readily find

$$M^3 = \sum_x \langle S_x^3 \rangle = M_0^3 - \sum_k \langle b_k^\dagger b_k \rangle = M_0^3 - \sum_k \frac{1}{e^{\beta(\omega_k - \mu)} - 1} \tag{38}$$

where $M_0^3 = N^3/2$ is the x^3-component of the magnetization at $T = 0$ (see Eq. 23). In the low-temperature limit, only low-energy modes contribute significant weight to the momentum sum; hence we approximate $\omega_k \approx B_0 + Ja^2|k|^2/2$.[2] We then proceed to solve Eq. 38 by working in the large-N limit (approximating the sum as an integral) and consider no external field or chemical potential;[2] the result is

$$M_0^3 - M^3 = N^3 \zeta(3/2) \left(\frac{k_B T}{2\pi J} \right)^{3/2} \tag{39}$$

where ζ is the Riemann zeta function. These details aren't important, what matters is that the physics the spin-wave model predicts is that at low-temperatures the magnetization should scale with temperature as

$$M_0^3 - M^3 \sim T^{3/2} \tag{40}$$

This is precisely what one finds experimentally for ferromagnets![2] This well known property is known as the "$T^{\frac{3}{2}}$ Bloch Law".[3] This provides strong evidence that, for temperatures far below the Curie temperature, spin-waves are indeed the emergent physics that describes a ferromagnet. The $T^{\frac{3}{2}}$ Bloch law was established long ago and since then spin-waves have been observed directly, thus confirming this notion: [3] contains a concise summary on the history of magnons and references experiments which established their existence.

Deviations from experiment will arise as the temperature increases though. One could try to remedy this by replacing the (approximate) magnon-gas dispersion with its exact form, but we also have to realize that as the temperature increases the $\mathcal{O}(b_x^4)$ terms we dropped in Eq. 24 start to become important.[2] So long as we remain in the regime of validity of the Holstein-Primakoff boson representation, $\langle b_x^\dagger b_x \rangle < 1$, we can improve our theory by expanding the square-root operator (which was set to unity in Eq. 21) to arbitrary order as

$$\sqrt{1 - b_x^\dagger b_x} = 1 - \frac{1}{2} b_x^\dagger b_x - \frac{1}{8}(b_x^\dagger b_x)^2 + \mathcal{O}(b_x^\dagger b_x)^3 \tag{41}$$

and then, after expanding the Holstein-Primakoff bosons in magnons, proceed to perform perturbation theory on the now interacting spin-wave Hamiltonian.[2] This approach is still limited in its usefulness however, because perturbation theory will take too long to converge at higher temperatures.[2]

5 Conclusion and Future Outlook

In the course of this essay we have explored the physics of magnetic materials by examining the quantum Heisenberg (minimal) model. We discussed the Holstein-Primakoff and Schwinger boson representations of spin operators and then proceeded to apply the Holstein-Primakoff transformation to determine the effective physics of the Heisenberg model in the ferromagnetic regime. We discovered the emergent physics of a ferromagnet at low temperatures is described by spin-waves or magnon quasiparticles; finally, we applied our formalism to compute the Magnetization of the system and found its temperature scaling consistent with experiment.

To conclude, we would like to briefly discuss the emerging scientific field of *magnonics* which seeks to exploit spin-waves for technological applications with the ultimate goal being to utilize magnons to "carry and process information on the nanoscale".[3] For this to become a reality, it will likely be imperative that such devices complement current semiconductor manufacturing processes in that they can be integrated onto circuits efficiently or that they possess capabilities not present in modern technologies.[3]

It has been shown that one can construct logic gates from magnonic devices — one example being with the so-called *magnetostatic spin-wave device*: this class of device is designed using input and output antennas connected to waveguides which can interact with a region of material harboring spin-waves.[3] However, in terms of meeting the criteria above, this approach seems unfavourable as the systems' length scale is large due to required external microwave circuits.[3] This is not the end of the story though; other designs have been (and are being) developed to implement logic using magnonic devices.[3] The field of magnonics is still relatively young and interesting developments are surely on the horizon.

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