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Selection, stability and renormalization

Lin-Yuan Chen, Nigel Goldenfeld, Y. Oono and Glenn Paquette[1]

Department of Physics, Materials Research Laboratory, and Beckman Institute,
1110 W. Green Street, University of Illinois at Urbana-Champaign, Urbana, IL 61801-3080,
USA

We illustrate how to extend the concept of structural stability through applying it to the front propagation speed selection problem. This consideration leads us to a renormalization group study of the problem. The study illustrates two very general conclusions: (1) singular perturbations in applied mathematics are best understood as renormalized perturbation methods, and (2) amplitude equations are renormalization group equations.

1. Introduction

When a very thin film made of diblock copolymers [1–3] in the disordered phase is quenched sufficiently, microphase separation occurs, and segregation patterns are formed. What happens if we cool the film from one end? We would expect the appearance of a segregation pattern invading the featureless disordered phase. The quenched film in the disordered state is thermo-dynamically unstable. Thus to facilitate the observability of such propagating front phenomena, the growth of spontaneous fluctuations before the front must be suppressed. This could be accomplished, for example, by sliding a cooling block along the film. If we slide the block too quickly or too slowly, however, we would not observe any intrinsic front invasion behavior into the disordered phase; if it is too fast, the unstable phase may spontaneously order before the front invasion, and if it is too slow, the invasion is restricted by the presence of the cooling front. What is the natural speed, given the quench depth? How does the pattern invade with this 'natural speed'? For example, suppose the equilibrium pattern for the low temperature state is a triangular lattice. When this phase invades into the quenched disordered phase, do we observe the triangular lattice immediately, or do we observe a lamellar phase first, which later orders into the triangular lattice? What are their speeds [4]?

Now, let us examine an example. Perhaps the simplest model of diblock copolymer melt dynamics is the following partial differential equation [5–8]:

[1] Present address: Department of Physics, Kyoto University, Kyoto 606, Japan.

$$\partial_t \psi = \Delta(-\tau\psi + g\psi^3 - D\,\Delta\psi) - B(\psi - \alpha)\,, \tag{1.1}$$

where ψ is the order parameter field, τ, g, D and B are positive constants, and α is a constant which could be negative. Fig. 1 illustrates the quenching process due to the moving cooling front simulated by the cell-dynamical system [9–11] corresponding to (1.1) [5]. In this particular case, lamellae parallel to the cooling front are first formed and then break up into a triangular pattern. In the steady state, a set of three modes, $\mathcal{W}_1 = \{W_{1,1}, W_{1,2}, W_{1,3}\}$, where each is parallel to one of the three edges of the triangle, invades the disordered region. In this illustration, the mode parallel to the cooling front, $W_{1,1}$, invades first,
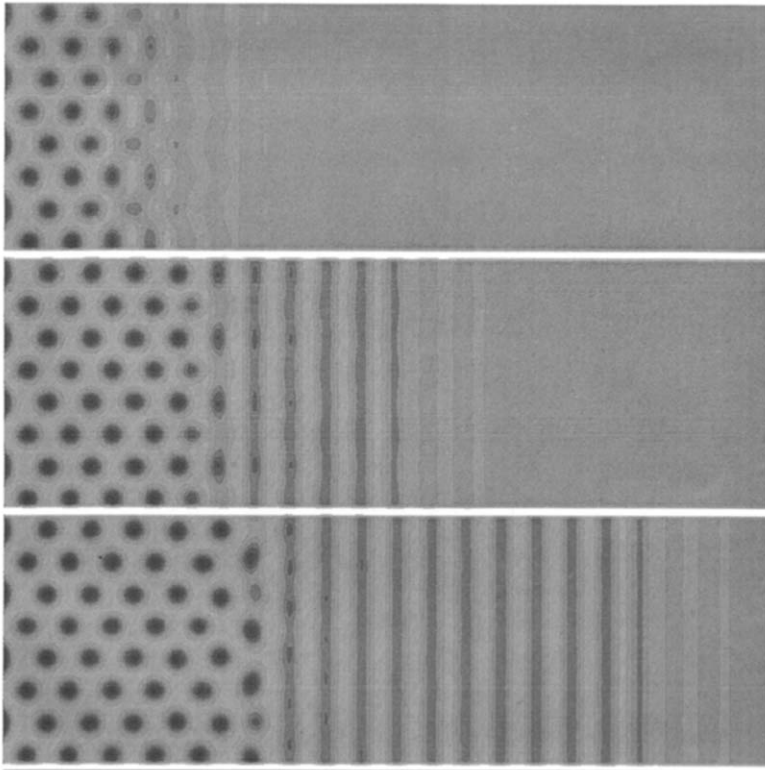

Fig. 1. A cell dynamics model simulation (for details, see [7]) of a block copolymer film with an invading triangular phase. Initially, a periodic pattern is imposed on the right edge of the system. As time proceeds, clearly the lamellar mode parallel to the invasion front leads the ordering process into the unstable uniform phase. Eventually the lamellar pattern breaks up into the final triangular pattern (defects may be introduced in this example because of a slight mismatching of the parameters and the system size). Thus, $W_{1,1}$ invades first, followed by the remaining two modes.

followed by the remaining two. Under the same boundary condition, but with different polymer parameters, sometimes a triangular lattice is formed by the invasion of the set $\mathcal{W}_2 = \{W_{2,1}, W_{2,2}, W_{2,3}\}$, which is rotated by 30 degrees with respect to $\mathcal{W}_1$. In general, prior to the establishment of a steady state 3-mode invasion, there is a competition between $\mathcal{W}_1$ and $\mathcal{W}_2$ (and any other modes which happen to be present). The time evolution of the invasion is governed by a set of simultaneous semilinear parabolic equations of the form [4]

$$\partial_t \varphi_i = D_i \, \Delta \varphi_i + F_i(\varphi_1, \ldots, \varphi_N) \qquad (i = 1, \ldots, N) \, , \tag{1.2}$$

where φ_i denotes the amplitude of the ith mode, N is the total number of relevant modes, D_i is the diffusion constant for the ith mode, and F_i is the 'reaction term' (a smooth function).

In this paper, we first wish to discuss the front selection problem for (1.2): when many stable propagating fronts are allowed by the model, what front can we actually observe under an ordinary experimental setting?

This question is, however, only the starting point of the present paper, whose main aim is to discuss and illustrate the fundamental role of re-normalization-group ideas in macroscopic physics.

The above question about selection has led us to the structural stability analysis [12] of (1.2) (section 3). A renormalized perturbation approach is given as an algorithm to check the observability criterion due to the structural stability analysis (section 4) [13]. This analysis leads us to a vast frontier of renormalization group theory (section 5) [14]. In section 2 we give a brief review of the selection problem. The last section contains a summary and comments. This article contains some pedagogical material to clearly demonstrate our points, but its main purpose is to announce an intimate relation among structural stability, renormalization and singular perturbation. More accurate and detailed statements will be published elsewhere.

2. Selection problem

The simplest case of (1.2) is obviously the following scalar equation:

$$\partial_t \varphi = \partial_x^2 \varphi + F(\varphi) \, . \tag{2.1}$$

Fisher introduced the equation with $F(\varphi) = \varphi(1 - \varphi)$ (Fisher's equation). We assume $F(0) = F(1) = 0$. If we also assume that $F(\varphi) > 0 \; \forall \varphi \in (0, 1)$, then

there exists a stable traveling wave solution interpolating between 1 and 0 with propagating speed c for all $c \in [c^*, +\infty)$. If F is differentiable at 0, then $c^* \geqslant \hat{c} \equiv 2\sqrt{F'(0)}$. Thus there are uncountably many stable propagating wave solutions for (2.1). However, usually only one of these is reproducibly observable in actual or computer experiments. Thus we have the selection problem: what stable traveling wave solution of (2.1) is actually observed?

To study the selection problem, we must carefully distinguish between the model and the system being modeled. We use the word 'system' to denote an actual physical system on which we can perform actual experiments. In contrast, a model is a mathematical procedure (or equation) describing the behavior of some observable(s) of the system which the model is to simulate. For example, the model (2.1) simulates a front propagation phenomenon such as the spreading of an allele of a gene locus in a population (the system). While the system apparently exhibits reproducibly a unique propagating front, the model allows uncountably many such fronts to exist. What is the selection rule for the propagating front solution which corresponds to the actually observed front in the system? This is the precise statement of the selection problem.

In an actual front propagation experiment, say, fire propagation along a fuse, we must prepare an initial condition. Fire is set by elevating the fuse temperature in front of the observer/experimenter. Thus, in practice the initial condition for the system is modified only on a finite region of the system. In the model, we must prepare the corresponding initial condition to have a compact support. Let us call such an initial condition a *physical initial condition*. We define the 'physical observability' (in the present context) of a solution to a given model as follows. If the traveling wave solution is attainable as an asymptotic state of the initial value problem with a physical initial condition, we call the traveling wave solution *physically observable*. This is sensible, since we cannot manipulate infinite space to prepare an initial condition. We can only modify the system just in front of us.

Aronson and Weinberger proved the following:

Theorem A (Aronson and Weinberger [15]). For (2.1) if $F(0) = F(1) = 0$, $F(x) > 0$ for any $x \in (0, 1)$, and if $F'(0) > 0$ (these conditions will henceforth be called the *AW condition*), then the boundaries of any level set for the value in $(0, 1)$ of the solution with a physical initial condition asymptotically travel with the speed c^*.

This implies that under the AW condition, the propagating speed we can actually observe is the minimum stable speed. We may call this the *minimum speed principle*. Empirically, this is what seems to be generally believed.

Certainly, we do not have any counterexample for (2.1), even without the AW condition. We do not, however, know any rigorous result other than this theorem.

There is a hypothesis of marginal stability due to Langer [16]. The linear marginal stability analysis is motivated by the following observation. Suppose a small localized perturbation is added to the $\varphi = 0$ state. Since this state is unstable, the disturbance grows, and consequently its fronts propagate in both directions. We wish to observe the front from a moving frame. If the speed of the frame is too slow, the disturbance front outruns us, so that we observe a growing disturbance and conclude that $\varphi = 0$ is unstable. If the speed of the frame is too fast, we outrun the disturbance, and we say $\varphi = 0$ is stable. However, the natural front should be self-sustained; the growth of the invading disturbance into the unstable state should be the cause of front propagation. Hence, the speed of the front should be the one which makes the $\varphi = 0$ state marginally stable.

In the moving frame with speed c, (2.1) reads

$$\partial_t \varphi = (\partial_\xi^2 + c\partial_\xi)\varphi + F(\varphi) , \tag{2.2}$$

where $\xi = x - ct$. We study the stability of the tip of the traveling wave in the following form:

$$\varphi = \epsilon(t)\, e^{k\xi} , \tag{2.3}$$

where ϵ is assumed to be very small. We get

$$\epsilon'(t) = \sigma(k)\, \epsilon(t) , \tag{2.4}$$

with

$$\sigma(k) = k^2 + ck + F'(0) . \tag{2.5}$$

The marginality condition is $\mathrm{Re}\,\sigma(k) = 0$ and $d\sigma(k)/dk = 0$. From these, we conclude that $c = 2\sqrt{F'(0)}$ is the selected speed according to the hypothesis. Notice that this value is a lower bound $\hat{c}$ for the minimum stable speed allowed to the model. Mathematically, we classify (2.1) into two cases [17]: if $c^* = \hat{c}$, the model is called a pulled case, and if $c^* > \hat{c}$, a pushed case. The linear marginal stability analysis works only when the model is pulled. There is no established method to distinguish pulled cases from pushed cases.

3. Structural stability

To motivate our approach to the selection problem, we first wish to reflect upon what we should mean by a good model of a natural phenomenon (or a given system).

Suppose we repeat the same experiment many times and collect data on the same observable for a given system. If the observed data cluster around some definite value, and the fluctuation around this value is small, we may say that the observable is reproducibly observable. Fluctuations around its most probable value are due to factors we cannot control. For example, they may be due to details in the initial condition or in the system preparation or maintenance itself. Now, let us assume that we have a mathematical model M of the system under study. If this is a good model of the system, then its behavior (at least that corresponding to the reproducible observables) must be stable against its modification. That is, in a certain sense, M is close to $M + \delta M$, where δM corresponds to the details beyond our control.

This is exactly the idea of 'structural stability' of a model first introduced in the context of dynamical systems by Andronov and Pontrjagin [18]. Since the coefficients of most differential equations important in practice (in physics, biology, engineering, etc.) cannot be determined exactly, it is crucial that their global features be largely unaffected by tiny changes in these coefficients. Therefore, Andronov and Pontrjagin proposed that only structurally stable models are good models to do scientific work. An epoch making theorem was later proven by Peixoto [19]: The set of all the structurally stable C^1-vector fields on a C^∞ compact 2-manifold is open and dense in the totality of C^1-vector fields. This was a very encouraging result, suggesting that we might dismiss all the structurally unstable models from science, as suggested by the original proposers of the concept[1]. However, soon it was recognized that if the dimension of the manifold is larger than 2, the structurally stable vector fields are not dense (see, for example, [20]).

What does this mean to science? It means at least that:

(i) the World is full of systems which are in a certain respect unstable and whose observable results are at least in part irreproducible.

Then, probably

(ii) the conventional definition of structural stability is too restrictive for science, since the fact that many things are not reproducible is reproducible.

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[1] An open and dense subset of a set should not be imagined to contain a 'majority' of the points in the set. For example, we can easily make an open dense subset of the interval $[0, 1]$ whose Lebesgue measure can be any small positive number, since there is a Cantor set whose measure is indefinitely close to unity.

If there are unstable or irreproducible aspects in the actual system being modeled, then a good mathematical model of the system must have features unstable with respect to the perturbation corresponding to that causing instability in the actual system. Thus a good model should be structurally stable with respect to the reproducibly observable aspects, but must be unstable with respect to the hard-to-reproduce aspects of the actual system.

Let us consider Fisher's equation

$$\partial_t \varphi = \partial_x^2 \varphi + \varphi(1 - \varphi) \,. \tag{3.1}$$

We wish to add δF to its 'reaction' term. If δF is C^1-small, that is, $|\delta F|$ is small and $|\delta F'|$ is also small in $[0, 1]$, then c^* changes only a little, and it is easy to demonstrate that actually all aspects of the model are structurally stable. That is, all changes are continuous with respect to the C^1-norm of δF. Unfortunately, it is easy to demonstrate that (3.1) is not stable against certain C^0-perturbations (i.e., without the smallness condition of $|\delta F'|$). Consider a small spine-like perturbation near the origin. Its size can be made indefinitely small while simultaneously making the slope of δF indefinitely large. Hence, we can indefinitely increase the slope of the reaction term at the origin with indefinitely C^0-small perturbations. This implies that the lower bound $\hat{c}$ of c^* can be increased without bound. Hence, the model cannot be structurally stable.

Is this an artifact of the mathematical model and thus a mere pathology? Consider the following analogy for (2.1). We may regard the equation to be describing the propagation of a flame along a fuse. In this analogy, φ is the temperature; 0 is the flash point of the fuse and 1 the steady burning temperature. The reaction term F may be regarded as the generation rate of heat due to burning. (Actually, it is the net rate of heat deposition on the fuse: the heat generation due to burning minus the loss of heat to the environment. In the steady state these must be the same, so $F(1) = 0$.) For $\varphi \approx 0$, we may linearize (2.1) as

$$\partial_t \varphi = \partial_x^2 \varphi + F'(0)\, \varphi \,. \tag{3.2}$$

If we put a very small amount of explosive powder along the fuse, we can increase $F'(0)$ considerably. The explosive burning near temperature 0 will therefore trigger a very fast propagation of fire along the fuse. Thus, we can imagine an actual system in which a drastic change of c^* is possible with a very small change of F. We may conclude that the structural instability of the model (2.1) is a desirable feature of a good model. This example thus provides an illustration of assertion (ii) above.

To relax the structural stability requirement of Andronov and Pontrjagin,

which requires every aspect[#2] of the model to be structurally stable, we must consider two things. First, we must require the stability of the model only against structural perturbations corresponding to perturbations of the actual system which affect its reproducible observables only slightly. We call such perturbations *physically small perturbations* of the system and the corresponding mathematical expressions *p-small perturbations* of the model. We require the structural stability of the model only against *p*-small perturbations. Secondly, we need not require every aspect of the model to be stable against *p*-small perturbations; we have only to require the stability of reproducibly observable features.

Our general conjecture is: solutions structurally stable against *p*-small perturbations describe reproducibly observable phenomena. More precisely, we conjecture a *structural stability hypothesis*: For a good model, only structurally stable consequences of the model are reproducibly observable. We must admit that there is potentially a tautology here. If we could reproducibly observe a phenomenon of a system which is not structurally stable in the model, or if we could not reproducibly observe something which the model says is structurally stable, then we conclude that the model is not a faithful picture of the system.

4. Structurally stable solutions of semilinear parabolic equations

For semilinear parabolic equations, we say a C^0-small perturbation is *p*-small if

$$\sup_{u \in (0,1]} \frac{\delta F(u)}{u} < f(\|\delta F\|_0) , \tag{4.1}$$

where $\| \cdot \|_0$ is the C^0-norm, and f is a continuous function such that $f(x) \to 0$ as $x \to 0$. Notice that the condition has no absolute sign, and only the upper bound of $\delta F/u$ is specified. Thus we are not demanding the differentiability of δF.

Now, we have the following theorem:

[#2] Although we said 'every aspect', this applies to the flow structure of the dynamical system. This does not mean that each trajectory is stable against small perturbations. Actually, as seen in Anosov systems, structurally stable systems are often highly chaotic. However, as is explicitly noted in [21], microscopic instability is the basis of macroscopic stability as exhibited in the pseudo-orbit tracing properties and the stability of invariant measures which minimize the topological pressure.

Theorem B (Paquette and Oono [12]). For (2.1) with $F(0) = F(1) = 0$, let $c^*(F)$ be the minimum traveling wave speed for the reaction term F. Then, if δF is p-small, $\lim_{\|\delta F\|_0 \to 0} c^*(F + \delta F) = c^*(F)$.

An intuitive idea behind theorem B is as follows. Suppose $\varphi(x, t) = \phi(\xi)$ (where $\xi \equiv x - ct$) is a traveling wave solution to (2.1). ϕ obeys

$$\frac{d^2\phi}{d\xi^2} + c\frac{d\phi}{d\xi} + F(\phi) = 0 , \tag{4.2}$$

or, replacing ϕ with q,

$$\dot{q} = p , \qquad \dot{p} = -cp - \frac{dV}{dq} , \tag{4.3}$$

where $F(q) = dV/dq$. That is, the problem can be interpreted as a particle of unit mass (position q and velocity p) sliding down a potential hill V with friction constant c. Hence in this particle analogy, the speed in the original problem corresponds to the friction constant.

A propagating front connecting 1 and 0 corresponds in the particle analogy to an orbit connecting the saddle S and the sink (at the origin) O, as shown in fig. 2. If c is too small, the particle overshoots 0 and goes into the region $q < 0$. The corresponding solution of the original partial differential equation is thus unstable in the ordinary sense of this word. As can be seen from fig. 2, c^* is the boundary between overdamped and underdamped motion. Now let us put a small potential bump at the origin; this can be done with an indefinitely C^0-small perturbation to F (or indefinitely C^1-small perturbation to V). Obviously, overdamped saddle–sink connection orbits no longer exist. That is, all the front solutions with speed faster than c^* are destroyed by this perturbation. Again obviously, sufficiently underdamped orbits still overshoot the origin, so that there must be a boundary between over and underdamped orbits which is not far away from the original c^*. For $c < c^*$, an appropriate bump would convert this c into the critical damping factor (that is, the minimum speed of the stable stationary front). However, in this case we can always choose a much smaller bump to leave c as an insufficient friction constant for the particle to stop at the origin. Hence, the boundary between over and underdamped cases must be infinitesimally close to the original c^*, if the perturbation is infinitesimally small.

This intuitive demonstration is technically not easy to rigorize, since allowed perturbations are not necessarily a simple bump. Still, it captures the salient physics (and mathematics) behind the structural stability of c^*.
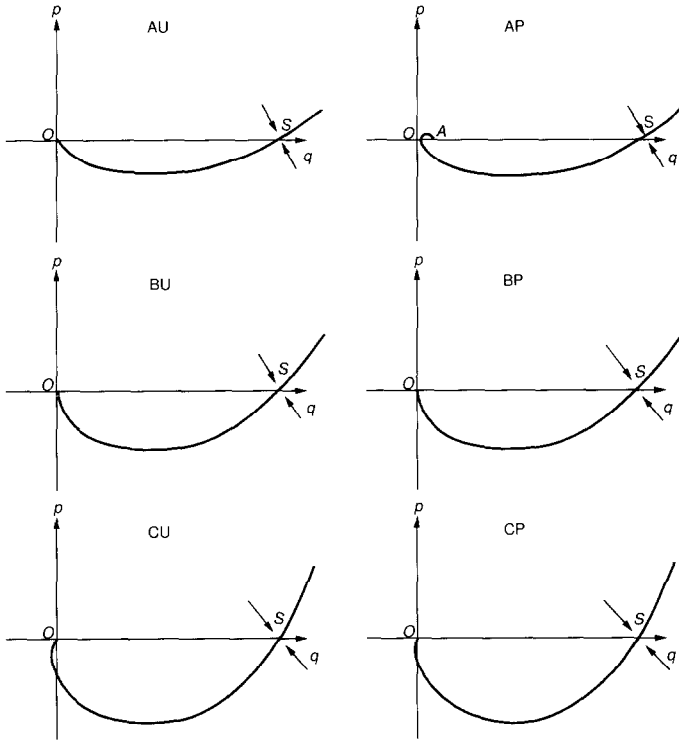
Fig. 2. An intuitive explanation of the structural stability of the slowest stable propagation speed
c^*. The trajectories corresponding to the traveling wave solutions are illustrated for (4.3). The left
column with U is for the unperturbed model, and the right column with P for the model perturbed
with a small potential bump at the origin. S is the saddle, and A is the newly formed stable point
with the potential bump. The friction constant c (that is, the front propagation speed in the original
problem) is decreased from A to C of the figure for both columns. BU illustrates the critical speed
c^* case; if c is slightly decreased further, then the trajectory overshoots the origin as CU. The
potential bump at the origin prevents all the overdamped trajectories like AU from reaching the
origin, as illustrated in AP. For most underdamped cases like CU, a small bump is not enough to
stop overshooting. Between AP and CP there must be a critical friction coefficient for the
perturbed model, but it must not be far away from the unperturbed one. Hence, c^* must be
structurally stable. Furthermore, no other c can be structurally stable.

If $q = 0$ is not an isolated minimum of V, the propagating solution of (2.1) is
unique. This can easily be seen from the particle analogy above. Notice that it
is always possible to eliminate the isolated minimum at $q = 0$ with a p-small
perturbation. This, together with theorem B, implies that c^* and only c^* is
structurally stable against physically benign perturbations.

In the present context, we accept that semilinear parabolic equations are
good models of front invasion into unstable states. Then the structural stability
hypothesis implies that the physically observable front speed is the minimum

stable speed. If the equation satisfies the AW condition, this is true thanks to Aronson and Weinberger's theorem A. But theorem B is valid even without this condition.

For the multimode case of (1.2), if $F_i = a_i \varphi_i$ + higher order terms, that is, if the φ are linearly decoupled, then we can prove a theorem analogous to theorem B [12]. In this case, however, structurally stable speeds need not be unique. Generally speaking, there is no further principle to select one among the structurally stable speeds. We believe that what we can observe in these cases depends on the initial condition. That is, only history can select the realized front among the structurally stable ones. Such examples have already been given, and in fact, the block copolymer model is one of these [12].

We have been unable to prove the general case where no linear decoupling assumption holds for (1.2). Still, we believe that what we have seen for the decoupled case holds here too (see example (5.1)). That is, what we can observe are structurally stable fronts, and only history can select the actually realized one among these.

Why does structural stability imply the minimum speed in this case? The key observation to explain this is that the speed $c > c^*$ is determined by the tip, while the speed c^* is determined by the bulk of the propagating front. The former may not be hard to understand, because to realize a speed faster than c^*, we need a fine tuning of the decay rate of the initial condition at infinity, as has been demonstrated in the pulled case by Kolmogorov et al. [22] and Kametaka [23]. For the pushed case, see [24]. The assertion that c^* is determined by the bulk may sound strange in the case of a pulled front, but it is easily seen that even in this case, c^* is insensitive to the tip. In both the pushed and pulled cases, note that if the initial condition is confined to a compact set, or decays to zero more quickly than any exponential, the resulting solution decays to zero more quickly than any exponential for all time. Also note that if the initial condition decays as $\sim \exp(-kx)$, where k is at least as large as k^* (here $\exp(-k^*x)$ is the asymptotic form of the steady state solution with speed c^*), then this asymptotic form is maintained for all time. In all of these cases, the asymptotic speed is c^*. The initial decay rate therefore determines the tip shape for all time, and hence this tip shape has nothing to do with the selected speed. Hence, the words 'pushed' and 'pulled' may both be misleading. (See [12] for a more detailed explanation.)

Now it is easy to understand why the minimum speed is structurally stable. Since the tip is extremely fragile against small modification of F near the origin, all speeds $c > c^*$ are unstable structurally. On the other hand, c^* is determined by the bulk of the propagating front, which is obviously insensitive to a small perturbation. In terms of the fuse analogy, imagine we put a thin film of water on the fuse. This would be sufficient to kill the fast propagation of

fire determined by the tip even if such propagation could be realized in the unperturbed system. Thus the structural instability of faster solutions is an actual phenomenon; that is, it is not an artifact of the modeling process. In this sense, the reaction–diffusion equation is a very good model of, e.g., the invasion of a stable phase into an unstable phase.

Since unstable states are unstable against spontaneous fluctuations due to, e.g., thermal fluctuations, it is not possible to prepare a wide unstable phase region. This is why the moving cooling front is used in the diblock copolymer example at the beginning of this paper. Therefore one might think that the nonuniqueness of the propagating front in the model is due to an excessive idealization of the actual system: the unstable state of the model is really a metastable state with a very small 'activation barrier'. One might conclude that this is the reason why in the actual system there is only one propagation speed which we observe. We need not deny that there are such cases, but in many actual examples, the unstable states are really unstable against some particular invasion mode, although they are metastable against spontaneous fluctuations.

Consider Fisher's original example of the spreading of an allele in a population. Of course, the invading allele could be produced *de novo* by mutation in the population, but this is extremely improbable, so the initial population is quite stable against spontaneous fluctuations. If the allele is advantageous, then the initial population is unstable against its invasion. In the case of the fuse analogy we have been using, the flash point T_F is the temperature at which the fuel becomes unstable against the invasion of radicals, while the ignition point T_1 is the temperature at which the fuel can spontaneously produce radicals (reacting with oxygen). That is, between T_F and T_1, the fuse is unstable against the invasion of fire, but metastable (almost stable) against spontaneous thermal fluctuations. The distinction between flash point and ignition point parallels the distinction between the secondary and primary nucleation processes. For example, a melt below the melting point should not be considered a metastable state when a crystal nucleus is already present. The melt is really unstable against the invasion of the crystal phase. Thus, the structural stability requirement cannot be regarded as simply an augmenting or auxiliary rule to make excessively idealized models realistic.

5. Renormalization and structural stability

Renormalization group (RG) methods are generally interpreted as a means to extract structurally stable features of a model [25,26]; the structurally stable features of the model characterize the universality class to which it belongs. In RG terminology theorem B implies that *p*-small perturbations are marginal

perturbations for c^*, but that some p-small perturbations are relevant to speeds larger than c^*. Furthermore, we know that generally speaking, C^0-small perturbations could be relevant.

Thus theorem B affords a method to judge whether the front with speed c_0 is observable or not through the study of its response to δF corresponding to a small potential bump added to the model: if the change of the speed δc vanishes in the limit of vanishing bump (that is, if δF is a marginal perturbation), then c_0 is observable. Otherwise, c_0 is not observable. As we have found, this procedure works numerically. In response to a p-small perturbation δF, the change in the speed of (1.2) observable in numerical computations vanishes with $\|\delta F\|_0$. Let us consider an example.

As stated above, we have been unable to prove a statement analogous to theorem B for multi-mode equations which display linear order coupling. We believe, of course, that our structural stability hypothesis applies to these equations as well, and in support of this conjecture, consider one such model for the present study. We note that similar behavior can also be easily observed for single-mode and multi-mode, linearly decoupled equations. Consider the following model equation:

$$\frac{\partial \psi_1}{\partial t} = 2\,\partial_x^2 \psi_1 + F_1(\psi_1, \psi_2)\,, \qquad \frac{\partial \psi_2}{\partial t} = \partial_x^2 \psi_1 + F_2(\psi_1, \psi_2)\,, \qquad (5.1)$$

where $F_1 = \psi_1 + \frac{1}{2}\psi_2 - \psi_1^3$, and $F_2 = 3\psi_2 + \frac{1}{2}\psi_1 - \psi_1^3 - \psi_2^3$. We numerically studied the behavior of (5.1) in response to the perturbation $F_1 \to F_1 + \delta F_1$ and $F_2 \to F_2 + \delta F_2$, where $\delta F_i = -10\psi_i$ if $\psi_i < \epsilon$ and 0 otherwise. Note that $(\delta F_1, \delta F_2)$ can be considered as the discretization of a p-small perturbation. $(\delta F_1, \delta F_2)$ is analogous to the film of water discussed in the context of the fuse analogy. If a traveling wave solution of (5.1) with speed c is observable (structurally stable), the speed of the observable solution of the perturbed equation must converge to c as $\epsilon \to 0$. We numerically determined the observable propagation speed of the unperturbed equation, as well as those of perturbed equations with several values of ϵ. The results of this study, shown in table I, support our structural stability hypothesis; the observable speed changes continuously in response to a p-small perturbation.

We next studied a "tip driven" solution (as opposed to the "bulk driven" solution considered above) of (5.1). We were able to produce such a solution by choosing two small positive values ϵ_1 and ϵ_2 to move at speed $c = 10$. We chose $\epsilon_1 = 0.248 \times 10^{-11}$ and $\epsilon_2 = 10^{-11}$. With these values, the eigenfunction of the linear equation corresponding to (5.1) for the traveling wave solution with $c = 10$ is given by: const. $\times (\epsilon_1, \epsilon_2) \exp(-kx)$, with $k = 0.323$. We then computed the speed of the resulting front by watching the point at which $\psi_i = 0.01$.

Table I
The observed speed of the front as a function of the size of perturbation. The speed is a continuous function of perturbation. That is, this observable speed is structurally stable.

δ	c	δ	c
10^{-5}	3.68	10^{-9}	3.81
10^{-6}	3.73	10^{-12}	3.83
10^{-7}	3.77	0	3.86
10^{-8}	3.79		

Not surprisingly, this value was 10. However, when we applied perturbations to the tip driven model identical to those applied to the bulk driven model, in each case, the propagation speed computed was also identical to that found for the bulk driven model. For the tip driven solution, the response of the model remains finite as the size of the perturbation vanishes. The above considerations thus lead us to conclude correctly that it is unobservable.

Once more returning to the propagation of fire as a physical analogy, this result can be interpreted as follows. For the dry fuse, we are able to force the system to exhibit 'fast' flame propagation by running a torch along the fuse to ignite it at the desired speed. When we add a film of water which the torch is not able to evaporate as it runs past, however, the behavior of this torched system cannot be distinguished from that of the untorched system. Its response to this small perturbation is therefore large.

Let ϕ_0 be a stable traveling wave solution of (2.1) with speed c_0. Let us add a p-small structural perturbation δF to (2.1) with $\|\delta F\|_0$ of order ϵ, and assume that in response the front solution is modified to $\phi_0 + \delta\phi$. Linearizing (2.1) to order ϵ in the moving frame with velocity c_0, we obtain formally the following naive perturbation result:

$$\delta\phi(\xi, t) = e^{-c_0\xi/2} \int_{t_0}^{t} dt' \int_{-\infty}^{+\infty} d\xi' \, G(\xi, t; \xi', t') \, e^{c_0\xi'/2} \, \delta F(\phi_0(\xi')) \,. \tag{5.2}$$

Here t_0 is a certain time before $\delta F(\phi_0(\xi))$ becomes nonzero, and G is the Green's function satisfying

$$\frac{\partial G}{\partial t} - \mathscr{L}G = \delta(t - t') \, \delta(\xi - \xi') \tag{5.3}$$

with $G \to 0$ in $|\xi - \xi'| \to \infty$, where

$$\mathcal{L} \equiv \frac{\partial^2}{\partial \xi^2} + F'(\phi_0(\xi)) - \tfrac{1}{4}c_0^2 \, . \tag{5.4}$$

Since by C^0-infinitesimally modifying F, we can always cause $\mathcal{L}$ to have 0 as an isolated eigenvalue, we may safely disregard all possible complications introduced by the presence of a 0 eigenvalue which is not isolated from the essential spectrum. Formally, G reads

$$G(\xi, t; \xi', t') = u_0(\xi) u_0^*(\xi') + \sum e^{-\lambda_n(t-t')} u_n(\xi) u_n^*(\xi') \, , \tag{5.5}$$

where $\mathcal{L}u_0 = 0$, and $\mathcal{L}u_n = \lambda_n u_n$. The summation symbol, which may imply appropriate integration, is over the spectrum other than the point spectrum $\{0\}$. Since the model is translationally symmetric, $u_0 \propto e^{c_0\xi/2}\phi_0'(\xi)$. Due to the known stability of the propagating wavefront, the operator $\mathcal{L}$ is dissipative, so 0 is the least upper bound of its spectrum. Hence, only u_0 contributes to the secular term (the term proportional to $t - t_0$) in $\delta\phi$. Thus we can write

$$\delta\phi_{\mathrm{B}} \equiv \delta\phi = -(t - t_0)\,\delta c\,\phi_0'(\xi) + (\delta\phi)_{\mathrm{r}} \, , \tag{5.6}$$

where the suffix B means "bare", $(\delta\phi)_{\mathrm{r}}$ is the bounded piece (regular part), and

$$\delta c = -\lim_{l \to \infty} \frac{\int_{-l}^{+l} d\xi\, e^{c_0\xi} \phi_0'(\xi)\, \delta F(\phi_0(\xi))}{\int_{-l}^{+l} d\xi\, e^{c_0\xi} \phi_0'^2(\xi)} \, . \tag{5.7}$$

One may immediately guess that this δc is the change in the front speed, but the naive perturbation theory is not controlled. A renormalization procedure can be used to justify the guess as follows [13].

The first term in (5.6) is divergent in the $t_0 \to -\infty$ limit. We introduce an arbitrary subtraction factor μ to separate the divergence by splitting $t - t_0$ as $t - \mu - (t_0 - \mu)$, and then absorb the divergence $\mu - t_0$ through renormalization of $\phi_0(\xi)$ to $\phi_{\mathrm{R}}(\xi, \mu)$. To order ϵ we get

$$\phi_{\mathrm{B}}(\xi) = \phi_{\mathrm{R}}(\xi, \mu) - \delta c(t - \mu)\,\phi_{\mathrm{R}}'(\xi, \mu) \, , \tag{5.8}$$

where ϕ_0 in the second term is replaced with ϕ_{R}, because δc is already of order ϵ, as seen from (5.7). The RG equation is $\partial\phi_{\mathrm{B}}(\xi)/\partial\mu = 0$. Hence, to order ϵ the RG equation is, after equating μ with t,

$$\frac{\partial\phi_{\mathrm{R}}}{\partial t} + \delta c\,\frac{\partial\phi_{\mathrm{R}}}{\partial \xi} = 0 \, . \tag{5.9}$$

Thus the speed of the renormalized wave is indeed $c_0 + \delta c$.

The formal expression (5.7) is legitimate only when both δF and $-\delta F$ are p-small. That is, the formula is legitimate only when δF is linearizable near the origin. Since we do not know whether the renormalized perturbation result is asymptotic or not, strictly speaking the formal expression (5.7) and the true change $\delta c \equiv c(F + \delta F) - c(F)$ itself should be distinguished. Furthermore, the expansion is correct only if the terms obtained are finite, so if c is not structurally stable, the formal expression may not be justified. Still, (5.7) seems to give us the correct information about the observability of c.

For example, if we add $\delta F = \epsilon\phi(1 - \phi)$ to (3.1) with $F = \phi(1 - \phi)$, then (5.7) gives $c^* \simeq 2 + \epsilon$; the exact result is, of course, $c^* = 2\sqrt{1 + \epsilon}$. If we add $\delta F = \theta(\phi - \Delta)(\phi - \Delta)(1 - \phi) - \phi(1 - \phi)$, with $\Delta > 0$ and θ being the unit step function, then $\delta c = \mathcal{O}(\sqrt{\Delta})$ for small Δ if $c_0 = 2$, and $\delta c \propto \sqrt{c_0^2 - 4}$ if $c_0 \geqslant 2$ in the $\Delta \rightarrow +0$ limit. Hence, only when $c_0 = 2$ does c change continuously with the perturbation.

6. Singular perturbation and renormalization

The reader may make the criticism that the renormalization approach in the preceding section is nothing but a singular perturbation approach (the method of stretched coordinate). Why do we need such a (purportedly) heavy machinery as RG? Before answering this question, we must stress that RG is not an esoteric machinery. As mentioned in the preceding section, it is a (the?) method to extract structurally stable features of a given model. For example, in the case of critical phenomena, we wish to study global features which are insensitive to small scale details. That is, we are pursuing the features of the model stable against structural perturbations corresponding to the small scale details.

In this section, we first demonstrate that the calculation in the preceding section is just the standard renormalization group theory for partial differential equations [14,26]. Then, we demonstrate that the ordinary singular perturbation method is understood very naturally from the RG point of view. Actually, we wish to claim that singular perturbations are most naturally understood as renormalized perturbations.

Introducing new variables $X = e^x$ and $T = e^t$, the propagating front solution reads $\phi(x - ct) = \Phi(XT^{-c})$. Thus the front speed is interpreted as an anomalous dimension. This is obvious; since the variables inside logarithms must be dimensionless, c cannot be determined by dimensional analysis. If we introduce T_0, defined by $t_0 = \ln T_0$, then $t - t_0 = \ln(T/T_0)$. From this we may

interpret T_0 as an "ultraviolet cutoff" scale. Hence, the $t_0 \to -\infty$ limit corresponds to the *cutoff* $\to 0$ limit in the usual field theoretic calculation or in our PDE calculation. In the ordinary multiplicative renormalization group scheme (see, for example, [27]), the logarithmic singularity $\ln(T/T_0)$ is absorbed into the renormalization group constants. Usually, we introduce an arbitrary length scale L and rewrite T/T_0 as $(T/L)(L/T_0)$. $\ln(L/T_0)$ is then removed by renormalization. Our μ above is nothing but $\ln L$, and the splitting of the logarithmic terms should correspond to the splitting $t - \mu + \mu - t_0$. $\mu - t_0$ represents the divergence to be absorbed into some phenomenological parameter.

Now, with the aid of the presumably simplest (but representative) example, we demonstrate our point that singular perturbation is best understood as renormalized perturbation. Consider the following linear ODE:

$$\epsilon \ddot{x} + \dot{x} + x = 0 . \tag{6.1}$$

We pretend that we cannot obtain its closed analytic solution and apply a very simple-minded perturbation approach. Expand x as $x = x_0 + \epsilon x_1 + \cdots$. We have

$$\dot{x}_0 + x_0 = 0 , \tag{6.2}$$

$$\dot{x}_1 + x_1 = -\ddot{x}_0 . \tag{6.3}$$

Solving these equations, we can easily get the following formal expansion:

$$x = A_0 \, e^{-(t-t_0)} - \epsilon A_0 (t - t_0) \, e^{-(t-t_0)} + \mathcal{O}(\epsilon^2) , \tag{6.4}$$

where A_0 is a constant determined by some initial condition. Now, the second term contains the prefactor $t - t_0$, and is thus a secular term; the ratio of the first and the zeroth order terms diverges in the $t_0 \to -\infty$ limit. As done above, we now introduce μ, split $t - t_0$ as $t - \mu + \mu - t_0$, and absorb $\mu - t_0$ into A_0, which is due to the initial condition we do not know. In this way, A_0 is renormalized to A. We rewrite (6.4) as

$$x = A \, e^{-(t-\mu)} - \epsilon A (t - \mu) \, e^{-(t-\mu)} + \mathcal{O}(\epsilon^2) . \tag{6.5}$$

Here A is a function of μ. Since μ is not in the original problem, obviously $\partial x / \partial \mu = 0$. This is the renormalization group equation. After differentiating (6.5) with μ and then setting μ equal to t, we get

$$\frac{\mathrm{d}A}{\mathrm{d}t} + A + \epsilon A = \mathcal{O}(\epsilon^2) . \tag{6.6}$$

This is exactly the equation obtainable, for example, by the reconstitution method [28]. Solving this equation (ignoring the second order term), and putting the result into (6.5) with $\mu = t$, we get

$$x = B\,e^{-(1+\epsilon)t}\,, \tag{6.7}$$

where B is the 'phenomenological constant' we must fix appropriately to reproduce the observable result. Clearly (6.7) is the formula obtained by the usual stretched coordinate method, or a multiscale expansion scheme. Here the result is obtained without the introduction of modified variables or coordinates.

One might think this agreement is only fortuitous. To see that this is not the case, consider (6.4) again. This formula is reliable if $\epsilon(t - t_0)$ is sufficiently small. Instead of calculating the result at t at once from t_0, we could proceed step by step just as in the Wilson renormalization group theory [29]. Let us divide t into N time spans and first solve the problem from 0 to t/N (for simplicity, we set t_0 to be 0). We get

$$x(t/N) = B\,e^{-t/N}(1 - \epsilon t/N) + \mathcal{O}((\epsilon/N)^2)\,. \tag{6.8}$$

Now use this as the initial value and solve $x(2t/N)$ to order ϵ, etc. We eventually get

$$x(t) = B[e^{-t/N}(1 - \epsilon t/N)]^N\,. \tag{6.9}$$

Taking the $N \to \infty$ limit, we get (6.7).

To obtain a solution reliable not only for large t but for all t, in the standard singular perturbation procedure, the so-called inner and outer expansion and their matching are required (see, for example, [30]). Now we demonstrate that from only the inner expansion, we can construct a uniformly valid solution by a renormalization group method.

First (6.1) is rewritten as

$$x'' + x' + \epsilon x = 0\,, \tag{6.10}$$

where $'$ implies the derivative with respect to $\tau \equiv t/\epsilon$. Naive perturbation gives the following result:

$$x = A_0 + B_0\,e^{-\tau} - \epsilon[A_0(\tau - 1 + e^{-\tau}) + B_0(1 - \tau\,e^{-\tau} - e^{-\tau})] + \mathcal{O}(\epsilon^2)\,. \tag{6.11}$$

Introducing μ into the secular terms through $\tau \to \tau - \mu + \mu$, we wish to absorb μ (here τ_0 is set to be 0 by an appropriate time shift) by renormalizing A_0 and

B_0. Let us proceed more systematically by introducing the multiplicative renormalization factors, $Z_A = 1 + \epsilon a_1 + \cdots$ and $Z_B = 1 + \epsilon b_1 + \cdots$, and renormalized coefficients as $A \equiv Z_A A_0$ and $B \equiv Z_B B_0$. Putting everything into (6.11), we get to order ϵ

$$
\begin{aligned}
x ={}& A(1 - \epsilon a_1 + \cdots) + B(1 - \epsilon b_1 + \cdots)\, \mathrm{e}^{-\tau} \\
& - \epsilon[A(\tau - \mu + \mu - 1 + \mathrm{e}^{-\tau}) + B(1 - (\tau - \mu + \mu)\, \mathrm{e}^{-\tau} - \mathrm{e}^{-\tau})] \\
& + \mathcal{O}(\epsilon^2)\,.
\end{aligned}
\tag{6.12}
$$

Thus the choice $a_1 = \mu$ and $b_1 = -\mu$ successfully eliminates the secular terms, and we get the renormalized perturbation result

$$
\begin{aligned}
x ={}& A + B\,\mathrm{e}^{-\tau} - \epsilon[A(\tau - \mu - 1 + \mathrm{e}^{-\tau}) + B(1 - (\tau - \mu)\, \mathrm{e}^{-\tau} - \mathrm{e}^{-\tau})] \\
& + \mathcal{O}(\epsilon^2)\,.
\end{aligned}
\tag{6.13}
$$

Notice that A and B are now functions of μ. Since x should not depend on μ, which is introduced independent of the original problem, we have the renormalization group equation $\partial x / \partial \mu = 0$. From (6.13) we get

$$
0 = \frac{\mathrm{d}A}{\mathrm{d}\mu} + \frac{\mathrm{d}B}{\mathrm{d}\mu}\, \mathrm{e}^{-\tau} - \epsilon(-A + B\,\mathrm{e}^{-\tau}) + \mathcal{O}(\epsilon^2)\,.
\tag{6.14}
$$

Here we have used the fact that derivatives are of order ϵ. Due to the functional independence of 1 and $\mathrm{e}^{-\tau}$, we get

$$
\frac{\mathrm{d}A}{\mathrm{d}\mu} = -\epsilon A\,, \qquad \frac{\mathrm{d}B}{\mathrm{d}\mu} = +\epsilon B\,.
\tag{6.15}
$$

Solving these and equating μ and τ in (6.13), we get

$$
x_R = A\,\mathrm{e}^{-\epsilon\tau} + B\,\mathrm{e}^{-(1-\epsilon)\tau} + \epsilon(A - B)(1 - \mathrm{e}^{-\tau})\,.
\tag{6.16}
$$

Let us compare this with the result obtained by the standard inner-outer matching method to order ϵ (that is, both the inner and outer solutions are obtained to order ϵ; notice that this calculation is partially second order):

$$
x = A\,\mathrm{e}^{-\epsilon\tau} + \beta\,\mathrm{e}^{-\tau} + B\epsilon\tau\,\mathrm{e}^{-\tau} + \epsilon(A - B)(\mathrm{e}^{-\epsilon\tau} - \mathrm{e}^{-\tau}) - \epsilon^2 A\tau\,\mathrm{e}^{-\epsilon\tau}\,.
\tag{6.17}
$$

Except for the ϵ^2 term, all the terms are correctly given by the RG procedure.

7. Reductive perturbation and renormalization

Now, let us look at (5.9). This is the equation of the wavefront as seen from the moving coordinate translating with the speed of the unperturbed front. From this frame the motion of the perturbed front is very slow. Hence, (5.9) is regarded as a slow-motion equation, like an amplitude equation obtained by the so-called reductive perturbation methods [31]. That an amplitude equation is an RG equation is not a fortuitous relation but a rule.

To see the point, let us consider the following slightly dissipative nonlinear hyperbolic equation:

$$\frac{\partial u}{\partial t} + \lambda(u) \frac{\partial u}{\partial x} = \eta \frac{\partial^2 u}{\partial x^2} , \qquad (7.1)$$

where $\lambda(u)$ is a sufficiently smooth function of u, and η is a positive constant. We consider a small amplitude wave in the background of the constant solution u_0,

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots , \qquad (7.2)$$

where ϵ denotes the amplitude of the wave.

First we study the case without dissipation ($\eta = 0$). Let us perform a naive perturbation approach. Let $\lambda_0 \equiv \lambda(u_0)$. We have

$$\partial_t u_1 + \lambda_0 \partial_x u_1 = 0 , \qquad (7.3)$$

$$\partial_t u_2 + \lambda_0 \partial_x u_2 = -\lambda'(u_0) u_1 \partial_x u_1 , \qquad (7.4)$$

and so forth. Introducing independent variables ($\xi \equiv x - \lambda_0 t, t$) to replace (x, t), these equations can be rewritten as (notice that ∂_t now reads $\partial_t + \lambda_0 \partial_x$)

$$\partial_t u_1 = 0 , \\ \partial_t u_2 = -\lambda'(u_0) u_1 \partial_\xi u_1 . \qquad (7.5)$$

Thus the right hand side of the second equation is a function solely of ξ in this coordinate system, so that it gives a secular term. Thus to order ϵ^2 we have the following general solution:

$$u = u_0 + \epsilon F_0(\xi) - \epsilon^2 (t - t_0) \lambda'(u_0) F_0(\xi) F_0'(\xi) . \qquad (7.6)$$

We introduce μ as we did for the propagation wave and split $t - t_0$ as

$t - \mu + \mu - t_0$. Then we absorb $\mu - t_0$ into the renormalized version $F(\xi, \mu)$ of $F_0(\xi)$. The renormalized perturbation result reads to order ϵ^2

$$u = u_0 + \epsilon F(\xi, \mu) - \epsilon^2 (t - \mu) \lambda'(u_0) F(\xi, \mu) \partial_\xi F(\xi, \mu) . \tag{7.7}$$

The renormalization group equation must be $\partial u / \partial \mu = 0$, so that we get to order ϵ^2

$$\partial_\mu F + \epsilon \lambda'(u_0) F \partial_\xi F = 0 . \tag{7.8}$$

If we identify μ and t in (7.7), we get $u = u_0 + F(\xi, t)$, so (7.8) with $\mu = t$, or

$$\partial_t F + \lambda_0 \partial_x F + \epsilon \lambda'(u_0) F \partial_x F = 0 , \tag{7.9}$$

in the original coordinate system is the equation of motion for the small amplitude wave.

With the introduction of a weak dissipation, the first equation of (7.5) should not be affected (this is the precise meaning of weak dissipation). At worst, only the second equation is modified as

$$\partial_t u_2 = -\lambda'(u_0) u_1 \partial_\xi u_1 + (\eta / \epsilon) \partial_\xi^2 u_1 . \tag{7.10}$$

Thus (7.6) is modified to be

$$u = u_0 + \epsilon F_0(\xi) - (t - t_0)[\epsilon^2 \lambda'(u_0) F_0(\xi) F_0'(\xi) + \epsilon \eta \partial_\xi^2 F_0(\xi)] . \tag{7.11}$$

Hence instead of (7.8), we arrive at the Burgers equation

$$\partial_t F + \epsilon \lambda'(u_0) F \partial_\xi F - \eta \partial_\xi^2 F = 0 . \tag{7.12}$$

This is of course a standard result obtained by a reductive perturbation method.

8. Summary

At the beginning of this paper, we illustrated how to generalize the concept of structural stability so that it is not excessively restrictive, and we applied it to the selection problem of front propagation speeds. Since the basic idea of renormalization group theory is to extract structurally stable features of a given model, this consideration naturally led us to the RG study of front propagation.

This study in turn revealed two very general conclusions, which are illustrated with simple examples:

(1) singular perturbation methods are best understood as renormalized perturbation methods, and

(2) amplitude equations are just RG equations.

The latter in particular strengthens our belief that RG is a prerequisite to do physics without being affected by unknown (high-energy) details of the world. A more systematic presentation with numerous examples (1) and (2) as well as the relations to the solvability condition, center manifold theory [32], etc. will be given elsewhere.

Acknowledgements

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