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Extreme fluctuations and the finite lifetime of the turbulent state

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We argue that the transition to turbulence is controlled by large amplitude events that follow extreme distribution theory. The theory suggests an explanation for recent observations of the turbulent state lifetime which exhibit superexponential scaling behavior with Reynolds number.

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The fundamental nature and stability of the turbulent state of fluids remains an open and challenging question. Fluid flow is characterized by a dimensionless number Re, which depends on the characteristic length L, velocity U, and kinematic viscosity of the fluid ν through the relation $\mathrm{Re} \equiv UL/\nu$. As the Reynolds number increases from zero, the flow becomes increasingly structured and eventually statistical in nature, and at large Re, the flow is said to be turbulent [1]. The conventional assumption—that the turbulent state is absolutely stable—has been challenged recently by a series of theoretical [2] and experimental probes [3–7] of the transition to turbulence. Taken as a whole, these works suggest that turbulence might, in some flow regimes at least, be a long-lived metastable state [2,8–11]. Such a view would be consistent with the fact that long-lived transient turbulent states can be excited as finite-amplitude instabilities of the laminar state so that the laminar and turbulent states can coexist (for a review of foundational work in this area, see, e.g., Ref. [12]; recent developments are summarized in Refs. [9–11,13]). However, the question remains as to whether the turbulent state is ever sustainable with an infinite lifetime for finite Reynolds numbers. This is a difficult experimental question to decide because the lifetime of the turbulent state can become so long that measurements become impossible. With the necessary restriction to a small range of Reynolds numbers, the data have, until recently, been difficult to interpret in a compelling way.

In a set of elegant and remarkably accurate experiments on transitional pipe turbulence [6], Hof *et al.* brought into question the idea that pipe flow turbulence is stable at long times beyond a finite critical Reynolds number [14,15]. The laminar state of a straight smooth pipe flow is linearly stable at all Reynolds numbers (see e.g., Ref. [16]), but a sufficiently large perturbation triggers localized turbulent puffs that persist for long times. The decay of the transient turbulent state is reported to follow a Poisson distribution, with a lifetime $\tau(\mathrm{Re})$ that increases sharply with increasing Reynolds number. The measurements of the lifetime of these localized puffs [6] reveal that $\tau(\mathrm{Re})$ apparently only diverges at infinite Reynolds number, scaling in a superexponential way with Re. Similar observations in another linearly stable flow—Taylor-Couette flow with outer cylinder rotation—have recently been reported by Borrero-Echeverry *et al.* [7].

In this Rapid Communication, we show that the form of the experimental data is consistent with a simple and general interpretation predicated on the use of extremal statistics. Our approach is related to the notion that the transient turbulent phenomena reflect escape from a low-dimensional dynamical attractor [17–19], but we conceive turbulence as a spatially extended phenomenon with a large number of degrees of freedom. The determining factor for the suppression of a puff is the probability that the largest fluctuation in a spatiotemporal interval consisting of multiple fluctuations fails to attain a threshold value. Thus we need to calculate the probability that the maximum amplitude of turbulent velocity fluctuations $\delta v(\vec{x},t)$ falls below some threshold value, which we term B_c. We will assume below that once the turbulence has been sufficiently suppressed, the turbulent state is quenched, an assumption consistent with previously published analyses [20–22]. Our calculation shows that the superexponential dependence of the lifetime of the turbulent state is a generic result of extremal statistics.

In order to understand the lifetime of turbulent puffs, we assume that turbulent velocity configurations may be regarded as independent beyond a correlation time τ_0 and that there is a probability p that the puff will be suppressed within each time interval τ_0. Then, the lifetime statistics will be Poisson. The probability P that turbulence persists to a time t after becoming established at a time t_0 is $P=(1-p)^M$, where the number of intervals is $M=(t-t_0)/\tau_0$. Therefore

$$\ln(P) = M\ln(1-p) = \frac{1}{\tau_0}(t-t_0)\ln(1-p), \tag{1}$$

and so it follows that $\tau_0/\tau=-\ln(1-p)$. Since $1 \gg p > 0$, we can estimate $\ln(1-p)=-p$ and therefore express the lifetime in the form

$$\tau = \tau_0/p, \tag{2}$$

where p depends on Re.

We now determine how p depends on the Reynolds number of the flow and potentially other factors. Within a spatial and temporal interval, multiple fluctuations occur, sampled from the turbulent velocity distribution $P_T(\delta v)$. The energy associated with these fluctuations is proportional to δv^2. We assume that when the energy fails to attain a certain threshold [20–22] at all points in the puff, the turbulent state becomes unstable and decays. Thus if the largest velocity fluctuation is less than the threshold, all of the turbulent fluctuations are less than the threshold. Accordingly, it is necessary to calculate the probability distribution of the maximum of the fluctuations within a puff, in order to ensure that the largest fluctuation is below the threshold.

We consider primarily a Gaussian distribution of velocity fluctuations but our arguments also apply to the case of an exponential distribution (as is the case at high Reynolds numbers). We seek the probability distribution $P_M(x)$ for the maximum x of a set of energy fluctuations $\{\delta v_i^2\}$, where $i=1,\ldots,N$. N represents the number of degrees of freedom and should scale with the size of the turbulent puff, denoted here by λ. Standard results from extreme statistics theory show that the appropriate result is the family of Fisher-Tippett distributions [23]. In particular, the universality class for P_M must be the type-I Fisher-Tippett distribution, sometimes known as the Gumbel distribution [24,25]

$$P_M(x) = \frac{1}{\beta}\exp[-(x-\mu)/\beta]\exp\{\exp[-(x-\mu)/\beta]\}, \quad (3)$$

where β sets the scale and μ the location of the distribution. Note that the scale and location will depend on N, because the maximum of a set of random variables will be an increasing function of the number of random variables. In particular, for the Gaussian case, Fisher and Tippett showed that asymptotically

$$\mu \sim \sqrt{\ln N}, \quad \beta \sim 1/\sqrt{\ln N}. \quad (4)$$

The mean and standard deviation of the Gumbel distribution are $\mu+\Gamma\beta$ and $\beta\pi/\sqrt{6}$, respectively, where $\Gamma\approx 0.577$ is the Euler-Mascheroni constant. The corresponding cumulative distribution is the probability that $x<X$ and is given by

$$F(X) \equiv \int_{-\infty}^{X} P_M(x)dx = \exp\{-\exp[-(X-\mu)/\beta]\}. \quad (5)$$

Thus, $p=F(B_c)$, where B_c is the threshold.

We anticipate that B_c is a decreasing function of Re, reflecting the intuition that at higher Re, turbulence can be more easily sustained by small fluctuations. We will consider the behavior of B_c as this sets the threshold in the distribution of energy maxima. The experiments are conducted in nominally smooth pipes within a narrow range of Re so it is appropriate to expand B_c about a particular Reynolds number Re_0, leading to $B_c=B_c^0+B_c^1(\mathrm{Re}-\mathrm{Re}_0)+O(\mathrm{Re}^2)$, where B_c^0 and B_c^1 are coefficients. In order to describe the same Reynolds number regime of the experiments, Re_0 may be interpreted to be a characteristic Reynolds number at which localized turbulent puffs first are observable so that the lifetime is order τ_0. This onset is not a precisely defined point, but for concreteness we will specifically define it to be the Reynolds number where $\tau=e\tau_0$. We will see that this choice simplifies the analysis below, but we emphasize that irrespective of whether or not we use the coefficient e or some other number of $O(1)$, our main predictions for the superexponential distribution are not affected. The freedom of choice in this definition of Re_0 is completely analogous to the arbitrariness in the definition of the coexistence point between liquid and gas, which is also dominated by nucleation phenomena, as has also been noticed by Manneville [22].

Collecting results, we find that the average lifetime of a turbulent puff will have the approximate functional form:

$$\tau = \tau_0 \exp[\exp\{-[B_c^0 + B_c^1(\mathrm{Re}-\mathrm{Re}_0) + O((\mathrm{Re}-\mathrm{Re}_0)^2)]\}] \quad (6)$$

in agreement with experimental findings. The coefficient τ_0 may in principle depend on pipe length or aspect ratio if these factors change the spatial scale on which regions of the localized turbulent puff are statistically independent and the time scale on which the state of the puff loses memory of previous states. The Taylor expansion only needs to be carried out to first order given the (understandably) small range of Reynolds number over which the experiments measuring the lifetime of turbulent puffs have been conducted. From Eq. (5) we can write

$$\ln\ln(\tau/\tau_0) = -(B_c-\mu)/\beta = c_1\,\mathrm{Re} + c_2, \quad (7)$$

where we have used the notation of Ref. [6] to denote the coefficients c_1 and c_2 of the linear fit to the data. Comparing with Eq. (6) we read off that

$$c_1 = -B_c^1/\beta \quad c_2 = -(B_c^0 - \mu - B_c^1\,\mathrm{Re}_0)/\beta. \quad (8)$$

Now, at Re_0, the lifetime becomes comparable to the correlation time τ_0 and to be concrete, we chose that for $\mathrm{Re}=\mathrm{Re}_0$, $\tau=e\tau_0$, although it is straightforward to verify that our results have only a very weak dependence on the precise coefficient used. Then from Eq. (6), we see that $B_c^0=\mu$ and thus the ratio of the coefficients $c_2/c_1=-\mathrm{Re}_0$. The physical interpretation, if any, of the fitting parameter Re_0 is not clear to us because although it is defined loosely as a characteristic Reynolds number below which the lifetime of the turbulent state is too small to be observable, any systematic dependence on intrinsic features of turbulence, flow geometry, and perhaps wall roughness is beyond the scope of our work and of the experiments available at this time. Moreover, the choice of definition of the Reynolds number in any particular geometry is not unique once multiple length scales are present, as in the case of Taylor-Couette flow, and thus it is hard to identify a unique and consistent definition of Re and Re_0 in order to retain meaning across different flow geometries.

We conclude with a comment about the dependence of τ on puff length λ. The number of degrees of freedom N active in the turbulent puff is proportional to λ. In our approach, the λ dependence of τ can be estimated by substituting the scaling of μ and β as given by Eq. (4) into our formula for τ. This yields that $\log(\tau)\propto\lambda$ to leading order, showing that the superexponential scaling with Reynolds number does not translate into a superexponential scaling with length and is consistent with numerical measurements reported in Ref. [21]. This prediction also applies if the probability distribution for velocity fluctuations is exponential because in this case $\mu\sim\log N$, i.e., as log λ, but β does not scale with N [23].

The interpretation of the lifetime statistics given here is related to that suggested recently by Manneville [22] and earlier considerations by Pomeau on nucleation phenomena in hydrodynamics [26]. Indeed, any spatially extended dynamical system with a memoryless subcritical bifurcation should be expected to yield extremal statistics, and we hope to report on detailed calculations of this phenomenon in a future publication [27].

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