# Testing a Missing Spectral Link in Turbulence

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Although the cardinal attribute of turbulence is the velocity fluctuations, these fluctuations have been ignored in theories of the frictional drag of turbulent flows. Our goal is to test a new theory that links the frictional drag to the spectral exponent $\alpha$, a property of the velocity fluctuations in a flow. We use a soap-film channel wherein for the first time the value of $\alpha$ can be switched between 3 and $5/3$, the two theoretically possible values in soap-film flows. To induce turbulence with $\alpha = 5/3$, we make one of the edges of the soap-film channel serrated. Remarkably, the new theory of the frictional drag holds in both soap-film flows (for either value of the spectral exponent $\alpha$) and ordinary pipe flows (where $\alpha = 5/3$), even though these types of flow are governed by different equations.

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Two aspects of turbulent flows have been the subjects of extensive, seemingly unrelated, research efforts: the frictional drag and the velocity fluctuations. The frictional drag [1,2] is a dimensionless form of the shear stress that a flow exerts on the wall that contains it. Dating back to the 1930s, the classical theory of the frictional drag [1,2] remains a mainstay of hydraulic engineering, used customarily to design canals, to determine the cost of pumping oil through a pipeline, and to ascertain the draining capacity of a river in flood, for example. The classical theory was predicated on dimensional analysis and similarity assumptions [1,2], without any reference to the velocity fluctuations, of which little would be known until the 1940s, following Kolmogorov's elucidation of the fabric of turbulence [1,3,4]. Kolmogorov conceived turbulence as an ensemble of swirling velocity fluctuations (or "eddies") in a broad spectrum of sizes, a type of conceptual model that harks back to the notebooks of da Vinci. The eddies carry turbulent kinetic energy, and Kolmogorov was able to predict that the allotment of this energy among eddies of different sizes (or wave numbers $k$) is described by the function $E(k) \propto k^{-\alpha}$, where the spectral exponent $\alpha$ should take the value $5/3$. The function $E(k)$, known as the energy spectrum of the velocity fluctuations, turned out to be readily measurable [4], and by 1962 Kolmogorov's prediction had been verified experimentally [5]. Since then, more has been learned about the velocity fluctuations (we now know, for example, that in many turbulent flows [6–8] the spectral exponent can take values other than $5/3$), and several variants of the classical theory of the frictional drag have been proposed [9,10]. But like the theory on which they have been patterned, all variants of the classical theory ignore the velocity fluctuations, and the spectral exponent plays no role in any of them. If there is a missing "spectral link" between the frictional drag and the velocity fluctuations, the classical theory and its variants are blind to it (and must be deemed incomplete).

In a resolute departure from the purview of the classical theory, the velocity fluctuations have been explicitly taken into account in a recent theory of the frictional drag [11]. In this new theory, for a turbulent flow on a smooth wall the functional relation between the frictional drag $f$ and the Reynolds number Re is mediated by the spectral exponent [12–14]:

$$f \propto \mathrm{Re}^{(1-\alpha)/(1+\alpha)}. \tag{1}$$

This is the spectral link expressed in mathematical form. The spectral link may be readily checked for pipe flows, where for ordinary fluids $\alpha = 5/3$. In this case, (1) becomes $f \propto \mathrm{Re}^{-1/4}$, the Blasius empirical scaling [2], which is known to be in excellent accord with the experimental data for ordinary pipe flows of moderate turbulent strength (starting from $\mathrm{Re} \approx 2500$ and up to $\mathrm{Re} \approx 100000$) [10].

In contrast to ordinary pipe flows and other flows in which $\alpha = 5/3$, two distinct values of the spectral exponent are theoretically possible in soap-film flows [15–17]: $5/3$ and 3. It follows that a stringent test of the spectral link might be carried out in a soap-film channel. In the standard soap-film channel [18], a soap film of thickness $\approx 10\ \mu$m hangs between two meter-long, vertical, mutually parallel wires spaced a few cm apart. Driven by gravity, the film tends to drain at the outlet of the channel. But the film is constantly fed a soapy solution at the inlet, and a steady

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flow soon becomes established within the film. It is easy to induce turbulence by piercing the film with the teeth of a comb. The result is a soap-film flow in which $\alpha = 3$, and in a previous work [14] we were able to verify experimentally that in such a flow $f \propto \mathrm{Re}^{-1/2}$, in accord with (1) for $\alpha = 3$. Other methods of inducing turbulence in the standard soap-film channel [17,19] have resulted in flows with mixed spectra [that is, spectra in which $E(k) \propto k^{-5/3}$ on a narrow range of wave numbers, whereas for larger wave numbers $E(k) \propto k^{-3}$]. Whether it is possible to realize a soap-film flow with a single spectral exponent of value 5/3 remains an open question.

Consider, however, a newly designed soap-film channel in which two vertical, mutually parallel, thin steel blades substitute for the wires of the standard soap-film channel. One of the blades has a straight edge in contact with the film, the other blade has a serrated edge in contact with the film. Thus, the new channel is bounded by a straight edge on one side and by a serrated edge on the other [Fig. 1(a)]. The straight edge acts on the soap-film flow as a smooth wall and will allow us to test the spectral link of (1), where $f$ is the frictional drag measured on a smooth wall. As for the serrated edge, recent computational simulations [13] indicate that the serrations could induce turbulence with $\alpha = 5/3$.

To assess the new soap-film channel, we compute [20] the energy spectrum $E(k)$ at several locations on the film from measurements carried out with a laser Doppler velocimeter (LDV). These measurements [20] consist of time series of the instantaneous velocity on the plane of the film. In Fig. 2(a) we show a set of energy spectra computed on a transect of the channel, from close to the straight edge (which corresponds to $y/w = 0$) to close to the tips of the serrations (which correspond to $y/w = 1$). From Fig. 2(a), it is apparent that the serrations induce turbulence with $\alpha = 5/3$—but not over the entire width of the channel. In fact, the value of $\alpha$ remains close to 3 over at least one-half of the width of the channel. Nevertheless, as we shall presently demonstrate, a flow in which $\alpha = 5/3$ over the entire width of the channel can be brought about by piercing the film with a cylindrical rod at the inlet, as sketched in Fig. 1(b). To understand the effect of the rod, in Fig. 2(b) we show a set of energy spectra computed on the centerline of the channel (which corresponds to $y/w = 0.5$) and downstream from the rod. As the distance from the rod increases, the spectral exponent lessens monotonically and reaches the value 5/3 about 15 cm downstream from the rod. The rod sheds vortices which grow in diameter as they are advected downstream. These vortices pick up the turbulence induced by the serrations and spread it across the width of the channel [Fig. 1(c)]. We find that in all of our experiments $\alpha = 5/3$ over the entire width of the channel on any transect that is at least 30 cm downstream from the rod [Fig. 2(c)].

Alternatively, the film may be pierced with the teeth of a comb, as sketched in Fig. 1(d). The teeth of the comb induce turbulence with $\alpha = 3$ (as they do in the standard channel). The turbulence induced by the serrations is swept aside [Fig. 1(e)], resulting in a flow in which $\alpha = 3$ over the entire width of the channel [Fig. 2(d)].

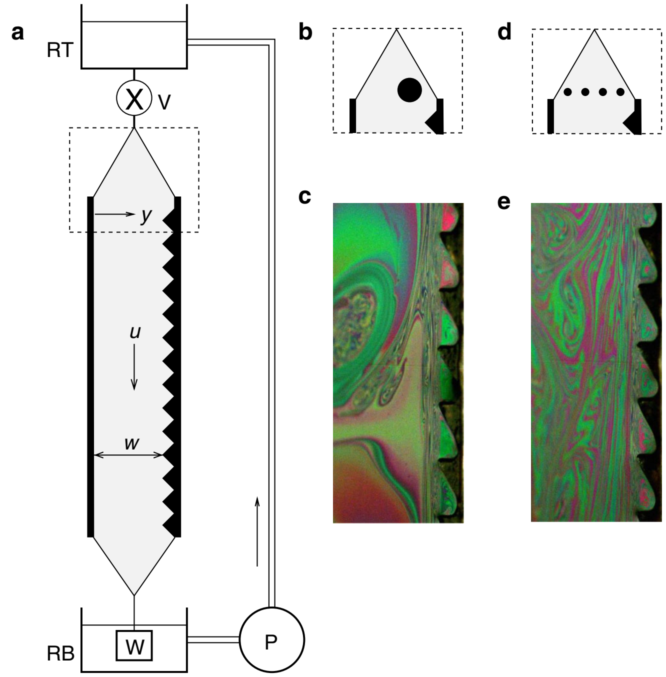


FIG. 1 (color online). The new soap-film channel. (a) The film (shown in gray shading) hangs between two steel blades (shown in black shading) that are suspended using fishing lines (shown as thin solid lines) kept taut by weight W. The spacing between the blades sets the width of the channel $w$, which we vary between 1 and 3 cm. The blades are 0.5 mm thick (in the direction normal to the plane of the figure), 1.2 m long, and $\approx 1$ cm wide. One of the blades has a serrated edge (size of the serrations $\approx 2$ mm; spacing between serrations $\approx 4$ mm). Reservoir RT contains a Newtonian soapy solution (2.5% of the commercial detergent "Dawn Nonultra" in water; kinematic viscosity $\nu = 0.01\ \mathrm{cm^2\,s^{-1}}$, the same as for pure water) [22], which flows through valve $V$ (which we use to control the flow rate) and into the film. The soapy solution is collected into reservoir RB and returned to reservoir RT through pump P. We carry our measurements of the frictional drag only on the straight edge of the channel, at distances of at least 30 cm and up to 85 cm from the inlet. (b) The film may be pierced with a cylindrical rod (diameter $\approx 6$ mm), resulting in a flow in which $\alpha = 5/3$ (see text); the turbulence can be made visible at a particular instant by (c) interference fringes in white light. (Note that the image does not cover the entire width of the channel.) (d) The film may be pierced with the teeth of a comb (tooth diameter $\approx 1$ mm; tooth spacing $\approx 3$ mm), resulting in a flow in which $\alpha = 3$; the turbulence can be made visible at a particular instant by (e) interference fringes in white light. (Note that the image does not cover the entire width of the channel.)

From LDV measurements, we also compute [20] the mean (time-averaged) streamwise velocity $u$ at any point on the film. Successive computations of $u$ along a transect of the channel give the mean velocity profile $u(y)$ of that

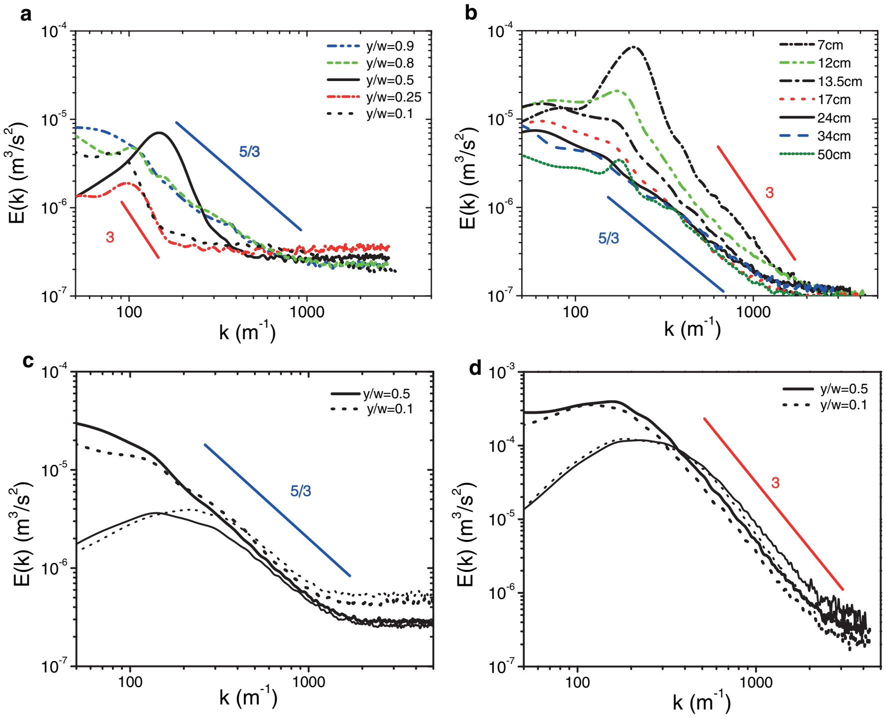


FIG. 2 (color online). Typical log-log plots of energy spectra computed from LDV measurements [20] carried out in the new soap-film channel. The slope of these log-log plots equals the value of the spectral exponent $\alpha$; unless otherwise noted, the plots correspond to the streamwise component of the energy spectrum, $E_{uu}(k_x)$. (a) Spectra on a transect of the channel [layout of Fig. 1(a)]. (b) Spectra on the centerline of the channel with a rod at the inlet [layout of Fig. 1(b)]. The distances indicated are measured downstream from the rod. (c) Spectra on a transect of the channel with a rod at the inlet [layout of Fig. 1(b)]. The transect is 30 cm downstream from the rod. The streamwise component $E_{uu}(k_x)$ is shown by thick lines, the transverse component $E_{vv}(k_x)$ is shown by thin lines; except for small $k$, both components collapse onto the same curve, signifying isotropy. (d) Spectra on a transect of the channel with a comb at the inlet [layout of Fig. 1(d)]. The transect is 30 cm downstream from the inlet. The streamwise component $E_{uu}(k_x)$ is shown by thick lines, the transverse component $E_{vv}(k_x)$ is shown by thin lines; except for small $k$, both components collapse onto the same curve, signifying isotropy.

transect. The mean velocity of the flow follows from the definition $U \equiv (1/w)\int_0^w u(y)dy$. To compute the viscous shear stress profile, $\tau_V(y)$, we use the formula $\tau_V(y) = \rho\nu du(y)/dy$, where $\rho$ and $\nu$ are the density and the kinematic viscosity of the soapy solution, respectively. We also compute [20] the Reynolds shear stress profile, $\tau_{\mathrm{Re}}(y)$. In Fig. 3 we show typical plots of $u(y)$, $\tau_V(y)$, and $\tau_{\mathrm{Re}}(y)$.

It is apparent from Fig. 3 that near the straight edge of the channel there is a viscous layer (of width $\approx 1$ mm) in which the Reynolds shear stress is negligible as compared to the viscous shear stress. From the slope $G$ of a mean velocity profile in the viscous layer (for example, inset of Fig. 3), we compute the shear stress between the flow and the straight edge as $\tau = \rho\nu G$. The frictional drag follows from the definition $f \equiv \tau/\rho U^2$, as $f = \nu G/U^2$. The attendant Reynolds number $\mathrm{Re} = wU/\nu$.

In Fig. 4 we show a log-log plot of the experimental data points ($f$, Re). In addition to the data points from the new soap-film channel, we include data points from the standard soap-film channel [14]. In accord with (1), the data points for $\alpha = 5/3$ (new channel only) are consistent with the scaling $f \propto \mathrm{Re}^{-1/4}$. Also in accord with (1), the data points for $\alpha = 3$ (new channel and standard channel) are consistent with the scaling $f \propto \mathrm{Re}^{-1/2}$. The frictional drag is inextricably linked to the energy spectrum of the velocity fluctuations.

To summarize, we have developed a new soap-film channel. In contrast with the standard soap-film channel, in which it is only possible to realize flows with a single spectral exponent of value 3, in the new soap-film channel it is also possible to realize flows with a single spectral exponent of value 5/3. We have used the new soap-film channel to confirm the theoretical prediction that there exists a link between the frictional drag and the spectral exponent (that is, the exponent of the energy spectrum of the velocity fluctuations). Some of the implications of this hitherto missing spectral link may be best appreciated by contrasting soap-film flows with pipe flows as follows.

Ordinary pipe flows are governed by the Navier-Stokes equations [1], whereas soap-film flows are acted upon by surface forces that are not considered in the Navier-Stokes equations, including elastic forces and the forces whereby the film interacts with the surrounding air [22,23]. Unlike pipe flows, soap-film flows are essentially two dimensional

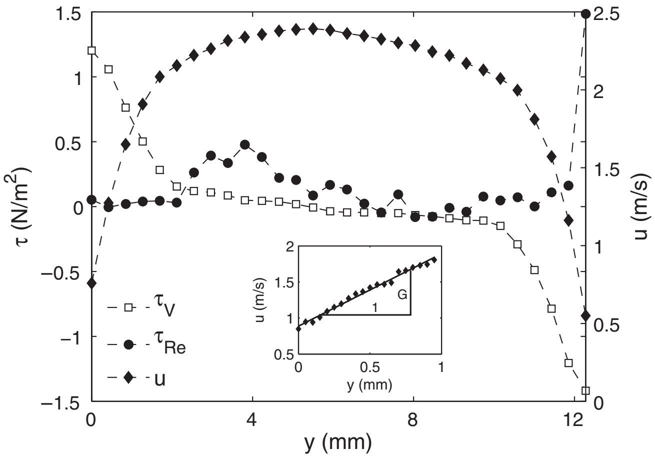


FIG. 3. The mean velocity profile $u(y)$, the Reynolds shear stress profile $\tau_{\mathrm{Re}}(y)$, and the viscous shear stress profile $\tau_v(y)$ of a typical soap-film flow with $\alpha = 5/3$. [Spectra of this flow are shown in Fig. 2(c).] Points on the film closer than $\approx 20\ \mu$m (the diameter of the beam of the LDV) from the straight edge of the channel cannot be probed with the LDV; thus, the first data point, which we position at $y = 0$, is at a distance of $\approx 20\ \mu$m from the straight edge of the channel. Inset: Zoom-in view of $u(y)$ in the viscous layer, where $du/dy = G$. The apparent slip velocity $u(0)$ can be ascribed to the complex flow immediately adjacent to the straight edge of the channel. In previous work [14] we have shown that the frictional drag does not depend on $u(0)$ (except perhaps through the Reynolds number).

(2D), and turbulence in two dimensions differs crucially from turbulence in three dimensions in that in two dimensions there can be no vortex stretching [1]. Further, even where a 2D flow has the same spectral exponent, $5/3$, as an ordinary pipe flow, the velocity fluctuations transfer

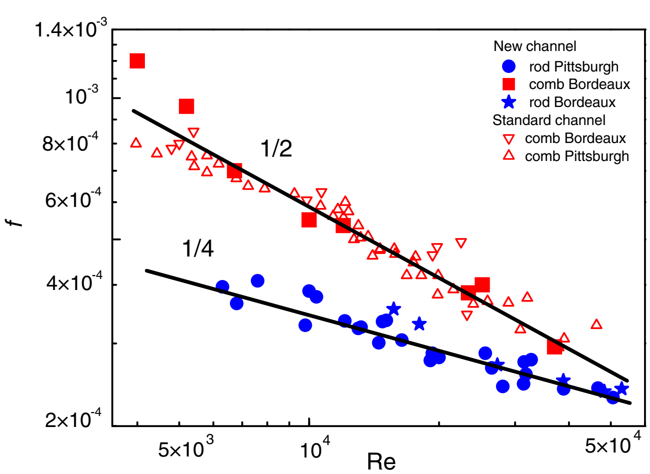


FIG. 4 (color online). Log-log plot of the experimental data points (frictional drag $f$, Reynolds number Re), from independent experiments carried out in Bordeaux and Pittsburgh. The data points for $\alpha = 5/3$ are shown using blue circles and stars, and the data points for $\alpha = 3$ are shown using red squares and triangles.

turbulent kinetic energy from smaller to larger length scales in the 2D flow, in the opposite direction in the pipe flow [15].

And yet, for all the profound disparities between soap-film flows and pipe flows, we have found that in both types of flow the relation between the frictional drag and the Reynolds number is set by the spectral exponent. Even where two flows have hardly anything in common, including the governing equations, besides the value of the spectral exponent, the relation between the frictional drag and the Reynolds number turns out to be the same in both flows, consistent with the existence of an inextricable, specific link between the frictional drag and the velocity fluctuations in a flow.

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[20] Experimental methods. We use a LDV with a sampling rate of 5 kHz to measure the two components of the instantaneous velocity at a point on the film, $u(t)$ (streamwise) and $v(t)$ (transverse). The LDV generates a pair of laser beams which converge on the film and scatter light from a spot 20 $\mu$m in diameter. The scattering is from submicron particles which we seed in the soapy solution. By carrying out measurements for about 200 s, we collect time series $u(t_i)$ and $v(t_i)$ and calculate the local mean velocities $u$ and $v$ as the time averages, $u \equiv \langle u(t_i) \rangle$ and $v \equiv \langle v(t_i) \rangle$, and the local Reynolds shear stress $\tau_{\rm Re}$ as the time average, $\tau_{\rm Re} \equiv \rho\nu\langle [u(t_i) - u][v(t_i) - v] \rangle$, where $\rho$ is the density of the soapy solution. To compute the local, streamwise component of the energy spectrum, $E_{uu}(k_x)$, we invoke Taylor's frozen-turbulence hypothesis [1] and carry out a space-for-time substitution $t \to x/u$ on the time series $[u(t_i) - u]$. This space-for-time substitution gives the space series, $u'(x_i) \equiv [u(x_i/u) - u]$, where $x_i = ut_i$. (The frozen-turbulence hypothesis is justified because in all our experiments the root mean square of the velocity fluctuations is less than 20% of $u$ [21].) The spectrum $E_{uu}(k_x)$ is the square of the magnitude of the discrete Fourier transform of $u'(x_i)$. The computation of $E_{vv}(k_x)$ is identical to the computation of $E_{uu}(k_x)$ except that $v$ substitutes for $u$.

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